8.1 Systems of linear equations

Concepts

Example: temperature distribution on a metal plate
Matrix notation
How many solutions can a linear system have?

Everywhere in applications of mathematics, you'll find systems of linear equations. Here's just one. Consider a square metal plate with edge 7 cm. Suppose some heating elements keep the temperatures along its edges at fixed values ranging from 0° to 6° as shown in Figure 8.1.1. For measurement, the plate is divided into 1- cm-square cells, and the temperature is regarded as constant within each cell. Edge cells have fixed temperatures, but the interior cells have variable temperatures $x_1, \ldots, x_{25}$ as shown. If no other heat sources are applied, the interior cell temperatures will approach steady state values. To compute these you can assume that in the steady state each of $x_1, \ldots, x_{25}$ is the average of the temperatures in its four neighboring cells. Thus, you can write a system of 25 linear equations, including

$$\begin{cases} x_1 = \frac{1}{4}\cdot 1 + \frac{1}{4}\cdot 1 + \frac{1}{4}x_2 + \frac{1}{4}x_6 \\ x_2 = \frac{1}{4}\cdot 2 + \frac{1}{4}x_1 + \frac{1}{4}x_3 + \frac{1}{4}x_7 \\ \quad\vdots \\ x_7 = \frac{1}{4}x_2 + \frac{1}{4}x_6 + \frac{1}{4}x_8 + \frac{1}{4}x_{12} \\ \quad\vdots \end{cases}$$

These can be rearranged as follows:

$$\begin{cases} x_1 - \frac{1}{4}x_2 - \frac{1}{4}x_6 = \frac{1}{2} \\ -\frac{1}{4}x_1 + x_2 - \frac{1}{4}x_3 - \frac{1}{4}x_7 = \frac{1}{2} \\ -\frac{1}{4}x_2 - \frac{1}{4}x_6 + x_7 - \frac{1}{4}x_8 - \frac{1}{4}x_{12} = 0 \\ \quad\vdots \end{cases}.$$

The solution $x_1, \ldots, x_{25}$ of this system of 25 equations is the steady state temperature distribution. Clearly, this method can be adapted for an arbitrarily shaped plate, and the system would become much larger if smaller cells were required for finer temperature resolution.

6.	5.	4.	3.	2.	1.	0.
5.	x_{21}	x_{22}	x_{23}	x_{24}	x_{25}	1.
4.	x_{16}	x_{17}	x_{18}	x_{19}	x_{20}	2.
3.	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	3.
2.	x_6	x_7	x_8	x_9	x_{10}	4.
1.	x_1	x_2	x_3	x_4	x_5	5.
0.	1.	2.	3.	4.	5.	6.

Figure 8.1.1 Temperature distribution on a metal plate

Most of this chapter is concerned with the general problem of solving a system of m linear equations

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$$

in n unknowns $x_1, \ldots, x_n$, where the coefficients a_{ij} and b_j are known scalars. (m and n can be any positive integers, not necessarily equal.) Matrix and vector algebra notation is generally used to abbreviate this system as $A\xi = \beta$, where A is an $m \times n$ matrix, and ξ and β are $n \times 1$ and $m \times 1$ column vectors:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \xi = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \beta = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

A linear system might have no solution at all, or exactly one, or infinitely many. For example, the system

$$\begin{cases} x_1 + 2x_2 = 3 & [1] \\ 4x_1 + 5x_2 = 6 & [2] \end{cases}$$

has a unique solution, determined as follows:

$$\begin{cases} x_1 + 2x_2 = 3 & [3] \\ \quad - 3x_2 = -6 & [4] = [2] - 4 \cdot [1] \end{cases}$$

$$\begin{cases} x_2 = 2 & [5] = [4] \text{ solved.} \\ x_1 = -1 & \text{Substitute } [5] \text{ into } [3] \text{ and solve.} \end{cases}$$

Systems with unique solutions are called *nonsingular*. Next, the system

$$\begin{cases} x_1 + 2x_2 = 3 & [6] \\ 4x_1 + 8x_2 = 12 & [7] \end{cases}$$

has more than one solution—for example, $x_1, x_2 = 0, {}^3/_2$ or $1, 1$. In fact, any solution of [6] also satisfies [7], so there are infinitely many solutions. Finally, this system obviously has no solution:

$$\begin{cases} x_1 + 2x_2 = 3 & [8] \\ 4x_1 + 8x_2 = 13 & [9]\,. \end{cases}$$

Section 8.2 presents the Gauss elimination method for solving nonsingular square systems. Section 8.4 implements that method using this book's MSP software. Section 8.5 discusses singular or nonsquare systems in detail.