

### 4.3 Fixpoint iteration

#### Concepts

Fixpoints and fixpoint iteration  
 Good and bad examples  
 Lipschitz constants  
 Conditions for convergence  
 Linear convergence  
 MSP function **Fixpoint**  
 Example: the cam function  
 Inverse functions

The problem of computing a solution of an equation is often transformed into one of finding a *fixpoint* (fixed point) of a function  $g$  —that is, of solving a special kind of equation  $x = g(x)$ . For example, solving  $x^2 = A$  for positive  $x$  and  $A$  is equivalent to solving  $x = g(x)$  for any of the three functions

$$g(x) = \frac{A}{x} \qquad g(x) = \frac{x + A}{x + 1} \qquad g(x) = \frac{1}{2} \left[ x + \frac{A}{x} \right]$$

You can sometimes compute a fixpoint by *fixpoint iteration*: guess an initial approximation  $x_0$ , then calculate successive approximations

$$x_1 = g(x_0), \quad x_2 = g(x_1), \quad \dots$$

If the sequence  $x_0, x_1, x_2, \dots$  approaches a limit  $p$  and  $g$  is continuous at  $p$ , then  $p$  is a fixpoint of  $g$  because

$$g(p) = g(\lim_{n \rightarrow \infty} x_n) = \lim_{n \rightarrow \infty} g(x_n) = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} x_n = p.$$

For the example functions and  $A = x_0 = 2$ , the fixpoint is  $p = \sqrt{2} \approx 1.414214$ . Fixpoint iteration yields the following approximations:

$$g(x) = 2/x \qquad g(x) = (x + 2)/(x + 1) \qquad g(x) = \frac{1}{2}[x + 2/x]$$

$n$	$x_n$	$n$	$x_n \approx$	$n$	$x_n \approx$
0	2	0	2.000000	0	2.000000
1	1	1	1.333333	1	1.500000
2	2	2	1.428571	2	1.416667
3	1	3	1.411765	3	1.414216
	<i>Bad!</i>		<i>Good</i>		<i>Excellent!</i>

Apparently, proper choice of  $g$  is vital! This choice and its consequences are analyzed next.

Suppose the fixpoint  $p$  and initial approximation  $x_0$  lie in a closed interval  $I = [a, b]$ . In order to apply  $g$  to the successive approximations  $x_1, x_2, \dots$  you may require that  $g$  map  $I$  into itself. Analysis of the convergence of sequence  $x_n$  involves the idea of a *Lipschitz constant* for  $g$  on  $I$ : a number  $L$  such that

$$|g(x) - g(x')| < L|x - x'|$$

for all  $x$  and  $x'$  in  $I$ . (This concept can be defined for functions with vector, matrix, and complex arguments or values, too: just replace the absolute value signs with appropriate norms.) Functions with Lipschitz constants are continuous, and Figure 4.3.1 shows that a continuous function  $g$  that maps  $I$  to itself has at least one fixpoint  $p$  there.

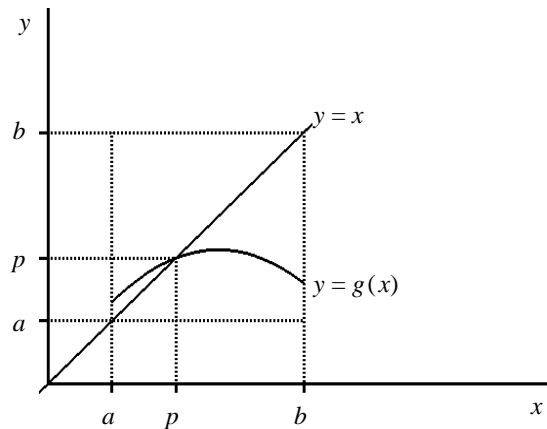


Figure 4.3.1  $g$  maps interval  $[a, b]$  to itself

If  $L < 1$ , there can be *only* one fixpoint: if  $q$  were another, then  $|p - q| = |g(p) - g(q)| < L|p - q| < |p - q|$ , contradiction!

Now, define the *error*  $\epsilon_n = x_n - p$  of the approximation  $x_n$ . Then

$$|\epsilon_{n+1}| = |x_{n+1} - p| = |g(x_n) - g(p)| \leq L|x_n - p| = L|\epsilon_n|,$$

and applying this inequality repeatedly yields

$$|\epsilon_n| \leq L^n |\epsilon_0|$$

for every  $n$ . If  $L < 1$ , then  $\epsilon_n \rightarrow 0$ , hence  $x_n \rightarrow p$ . The type of convergence displayed here has a special name: if a sequence  $x_0, x_1, x_2, \dots$  approaches a limit  $p$  and successive error terms  $\epsilon_n = x_n - p$  are related by an inequality  $|\epsilon_{n+1}| \leq L|\epsilon_n|$  with a constant  $L < 1$ , then  $x_n$  is said to converge to  $p$  *linearly*. (This concept can be extended to vector, matrix and complex sequences by using norms.)

This analysis has demonstrated that you can use fixpoint iteration to approximate a fixpoint  $p = g(p)$  if  $g$  maps a closed interval  $I$  containing  $p$  into itself, has a Lipschitz constant  $L < 1$  there, and your initial guess lies in  $I$ . The successive approximations converge to  $p$  linearly. Moreover, suppose you know a bound  $E > |\epsilon_0|$  for the error in the initial approximation. Then you can use the inequality  $|\epsilon_n| \leq L^n |\epsilon_0|$  to estimate the number  $n$  of iterations required to ensure that the error is less than a specified tolerance  $T$ : stop when  $L^n E < T$  —that is,

$$n > \frac{\log(T/E)}{E \log L} .$$

This analysis partially explains the difference between the three functions  $g$  suggested at the beginning of this section to solve the equation  $x^2 = A$ . For the first one,  $g(x) = A/x$ , the required inequalities  $g(a) \leq b$  and  $a \leq g(b)$  would imply  $A \leq ab \leq A$ , hence  $A = ab$  and  $g(a) - g(b) = b - a$ . Thus, there's no Lipschitz constant  $L < 1$  for any interval  $[a, b]$  that  $g$  maps to itself. The condition for using fixpoint iteration fails, and the successive approximations don't converge. This function won't be considered further.

The difference between the good and excellent results given earlier for fixpoint iteration with the other two functions

$$g(x) = \frac{x + A}{x + 1} \qquad g(x) = \frac{1}{2} \left[ x + \frac{A}{x} \right]$$

stems from the values of their derivatives at the fixpoint  $p = \sqrt{A}$ . If  $g'$  is continuous on the interval  $I$  under analysis and  $L$  is a number such that  $|g'(\zeta)| < L$  for all  $\zeta$  in  $I$ , then  $L$  is a Lipschitz constant for  $g$  on  $I$ : for each  $x$  and  $x'$  in  $I$ , the mean-value theorem provides  $\zeta$  between  $x$  and  $x'$  such that

$$|g(x) - g(x')| = |g'(\zeta)(x - x')| \leq L|x - x'| .$$

In fact, a closer analysis [7,Section 3.3] shows that if  $g'$  is continuous at  $p$  and  $|g'(p)| < L < 1$ , then you can find some closed bounded interval  $I$  containing  $p$  that  $g$  maps into itself, for which  $L$  is a Lipschitz constant. Thus you can use fixpoint iteration to approximate a fixpoint  $p = g(p)$ , provided  $|g'(p)| < 1$  and your initial approximation  $x_0$  is close enough to  $p$  to lie in  $I$ . (Unfortunately, this analysis doesn't tell *how* close you must be.) Further, you can use  $L$  with the inequality  $|\varepsilon_n| \leq L^n |\varepsilon_0|$  to estimate the number of iterations required.

Here are the derivatives of the two functions  $g$  still under consideration:

$$\begin{aligned}g(x) &= \frac{x+A}{x+1} & g(x) &= \frac{1}{2} \left[ x + \frac{A}{x} \right] \\g'(x) &= \frac{1-A}{(1+x)^2} & g'(x) &= \frac{1}{2} \left[ 1 - \frac{A}{x^2} \right] \\g'(\sqrt{A}) &= \frac{1-\sqrt{A}}{1+\sqrt{A}} < 1 & g'(\sqrt{A}) &= 0.\end{aligned}$$

For  $A = 2$ , the left hand function has  $g'(\sqrt{A}) \approx -0.17$ ; so  $L = 1/5$  is a reasonable Lipschitz constant, and the error  $\varepsilon_n$  will decrease by at least a factor of 5 with each successive approximation. The right hand function has  $g'(\sqrt{A}) = 0$  for any  $A$ ; hence *any* positive  $L$ , no matter how small, can serve as a Lipschitz constant. This function is clearly the best choice for solving  $x^2 = A$  by fixpoint iteration. It displays a phenomenon called *quadratic convergence*, which will be considered in Section 4.4.

Implementing fixpoint iteration requires a criterion for stopping the process. A client could find a Lipschitz constant and an initial error estimate and then determine the number of iterations required. But that's unrealistic, because those values are often difficult to estimate. The usual criterion is to stop when the distance between an approximation  $x$  and the corresponding function  $g(x)$  becomes less than a specified tolerance  $T$ . Since that's not guaranteed to happen, the implementation must include an upper limit on the number of iterations.

The MSP implementation is function **Fixpoint**, shown in Figure 4.3.2. It has parameters corresponding to the function  $g$ , the initial approximation  $x$ , and the tolerance  $T$ . It's implemented by a template for scalars of any type, because fixpoint iteration works with complex functions as well as real. An additional reference parameter **Code** reports the outcome of the computation: **Code** = 0 indicates that the iteration terminated satisfactorily

with  $|x - g(x)| < T$ , whereas **Code** = -1 signifies that this condition was not attained even after one thousand iterations. (This upper limit was chosen arbitrarily.)

```

template<class Scalar>
    Scalar Fixpoint(Scalar g(Scalar), // Find a fixpoint of g .
                   Scalar x,        // Initial estimate.
                   double T,        // Tolerance.
                   int& Code) {     // Status code.
    try {
        int n = 0; // Iteration count.
        Scalar p,dx;
        do { // Fixpoint
            p = g(x); // iteration:
            dx = x - p; // do <= 1000
            x = p; } // steps. Re-
            while (++n < 1000 && abs(dx) >= T); // port success,
        if (abs(dx) < T) Code = 0; // or failure to
            else Code = -1; // converge.
        return x; }
    catch(...) {
        cerr << "\nwhile executing Fixpoint ; now x,T = "
            << x << ', ' << T;
        throw; }}
    
```

Figure 4.3.2 Equate1 function Fixpoint

As a more realistic example to demonstrate **Fixpoint**, consider the cam function  $f(x)$  of Figure 4.2.2. The equation  $f(x) = 0$  is equivalent to

$$\frac{1 + e^{-\frac{x}{2\pi}} \sin x}{2} = 0.7$$

$$e^{-\frac{x}{2\pi}} \sin x = 0.4$$

$$\sin x = 0.4 e^{\frac{x}{2\pi}}$$

$$x = \arcsin\left(0.4 e^{\frac{x}{2\pi}}\right) = g(x) .$$

Thus, the root of the cam function is the fixpoint of this function  $g$ . Figure 4.3.3 shows execution of function **Fixpoint** to locate it.

**Finding the fixpoint  $x$  of  $g(x) = \text{asin}(.4 \exp(x/2\pi))$   
 by Fixpoint iteration**

**Initial estimate  $x$  : 0**  
**Tolerance T : .000001**

<b>Iteration</b>	<b>x</b>	<b>g(x)</b>	<b>x - g(x)</b>
0	0.0	0.41	-0.41
1	0.41	0.44	-0.030
2	0.44	0.44	-0.0022
3	0.44	0.44	-0.00017
4	0.44	0.44	-1.3e-05
5	0.44	0.44	-9.7e-07

**Fixpoint  $x = 0.4436807$**   
**Code = 0**

**Figure 4.3.3 Executing function **Fixpoint****

In Figure 4.3.3, the ratios of successive values of  $x - g(x)$  are approximately constant,  $\approx 0.075$ . There's a reason for that:

$$\lim_{n \rightarrow \infty} \frac{x_{n+1} - g(x_{n+1})}{x_n - g(x_n)} = \lim_{n \rightarrow \infty} \frac{g(x_n) - g(x_{n+1})}{x_n - x_{n+1}} = \lim_{n \rightarrow \infty} g'(\xi_n) = g'(p).$$

In this equation,  $\xi_n$  is a point (provided by the mean-value theorem) between the approximations  $x_{n+1}$  and  $x_n$ . Now differentiate  $g$ , compute  $g'(p) \approx 0.076$  for the fixpoint  $p$ , and thus explain the values of the ratios.

Earlier, you saw that one of the conditions for successful fixpoint iteration is that  $|g'(p)| < 1$ , where  $p = g(p)$  is the fixpoint. The following ratio limit, similar to the one just computed, shows what happens when  $|g'(p)| > 1$ . If  $g'$  is continuous in a neighborhood of  $p$ , then

$$\lim_{n \rightarrow \infty} \frac{\varepsilon_{n+1}}{\varepsilon_n} = \lim_{n \rightarrow \infty} \frac{g(x_{n+1}) - p}{g(x_n) - p} = \lim_{n \rightarrow \infty} \frac{g(x_{n+1}) - g(p)}{x_{n+1} - p} = \lim_{n \rightarrow \infty} g'(\xi_n) = g'(p).$$

Here,  $\xi_n$  lies between  $x_{n+1}$  and  $p$ . If  $|g'(p)| > 1$ , then  $|\varepsilon_n|$  increases without bound: the sequence  $x_n$  diverges.

An inverse function  $g^{-1}$  has the same fixpoints as  $g$ :  $p = g(p)$  if and only if  $g^{-1}(p) = p$ . If these functions are differentiable, then their derivatives at  $p$  are reciprocal. Except in the rare case when both have absolute value 1, one will have absolute value  $< 1$ , and the other,  $> 1$ . Thus, you'll be able to use fixpoint iteration with *exactly one* of them to find  $p$ . For example, the inverse of the Figure 4.3.3 function  $g$  is  $g^{-1}(x) = 2\pi \log(2.5 \sin x)$ . If you perform fixpoint iteration on  $g^{-1}$  starting with the final approximation  $x_5 = 0.4436807$  in the figure, you'll just get the original values  $x_4, x_3, \dots$  in reverse order. The sequence diverges.