

TAYLOR'S THEOREM

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Taylor's theorem plays a central role in numerical analysis, providing a method for approximating arbitrary functions by polynomials, and estimating the errors. There are several common proofs. The one that follows, from [James WOLFE, "A proof of Taylor's formula," *American mathematical monthly*, 60 \(1953\), 415-416](#), is appropriate because it depends on an extension of Rolle's theorem, like some other results fundamental to this subject.

According to Rolle's theorem, if a function g is differentiable on an open interval $I = (a, b)$, continuous at a and b , and $g(a) = g(b) = 0$, then $g'(\xi) = 0$ for some ξ in I . The extension needed here is concerned with higher derivatives:

if g has n continuous derivatives on the closed interval $[a, b]$, $g^{(n+1)}$ exists on its interior I , $g^{(j)}(a) = 0$ for $j = 0$ to n , and $g(b) = 0$, then $g^{(n+1)}(\xi) = 0$ for some ξ in I .

The proof is simple. First, g satisfies the hypotheses of Rolle's theorem on the interval $[a, b]$, so $g'(x_1) = 0$ for some x_1 in I . Now apply the same argument to g' on the interval $I_1 = [a, x_1]$ to determine x_2 in I_1 with $g''(x_2) = 0$. Continue the process until you've found $\xi = x_{n+1}$ in I_n with $g^{(n+1)}(x_{n+1}) = 0$.

With this preliminary step completed, you can turn to approximation by polynomials. Consider, as an example, a fourth degree polynomial and its derivatives:

$$\begin{aligned}P(x) &= P^{(0)}(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + a_4(x-a)^4 \\P'(x) &= P^{(1)}(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + 4a_4(x-a)^3 \\P''(x) &= P^{(2)}(x) = 2a_2 + 2 \cdot 3a_3(x-a) + 3 \cdot 4a_4(x-a)^2 \\P^{(3)}(x) &= 2 \cdot 3a_3 + 2 \cdot 3 \cdot 4a_4(x-a) \\P^{(4)}(x) &= 2 \cdot 3 \cdot 4a_4\end{aligned}$$

This should convince you that, in general, for an n th degree polynomial

$$P(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \dots + a_n(x-a)^n \quad (*)$$

the formula

$$P^{(j)}(a) = j! a_j$$

holds for $j = 0$ to n . You can use it in reverse: to construct a polynomial (*) with prescribed derivatives $P^{(j)}(a) = p_j$, set $a_j = p_j/j!$ for $j = 0$ to n .

Now let f be any function with n derivatives at a point a . Use the method of the previous paragraph to construct a polynomial $T_{n,a}(x)$ with the same derivatives there:

$$T_{n,a}(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \dots + a_n(x-a)^n$$

$$\left. \begin{aligned} a_j &= \frac{f^{(j)}(a)}{j!} \\ T_{n,a}^{(j)}(a) &= f^{(j)}(a) \end{aligned} \right\} \text{ for } j = 0 \text{ to } n.$$

$T_{n,a}$ is called the n th degree *Taylor* polynomial for f at a . When $a = 0$, it's also called a *Maclaurin* polynomial.

If f is sufficiently smooth, you can approximate values $f(x)$ by those of Taylor polynomials even at arguments x distant from a . *Taylor's theorem* lets you determine how good an approximation it is:

Suppose f has n continuous derivatives on the closed interval $[a, b]$, and the $n+1$ st exists on its interior I . Then there exists ξ in I such that

$$f(b) = T_{n,a}(b) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(b-a)^{n+1}.$$

The right hand term is called the *error term*: it tells how far the Taylor polynomial deviates from the function value at x . There are several versions of Taylor's theorem with different but equivalent error terms; this one is known as *Lagrange's*. If you can estimate the size of $f^{(n+1)}(\xi)$, then you can estimate the accuracy of the Taylor approximation. Note that Taylor's theorem for $n = 0$ is simply the mean-value theorem.

To follow Wolfe's proof, define

$$k = \frac{f(b) - T_{n,a}(b)}{(b-a)^{n+1}}$$

$$g(x) = f(x) - T_{n,a}(x) - k(x-a)^{n+1}.$$

for all x . Then for $j = 0$ to n ,

$$g^{(j)}(a) = f^{(j)}(a) - T_{n,a}^{(j)}(a) - k \left. \frac{d^j}{dx^j} (x-a)^{n+1} \right|_{x=a}$$

$$= f^{(j)}(a) - f^{(j)}(a) - k \cdot 0 = 0.$$

Moreover, $g(b) = f(b) - T_{n,a}(b) - k(b-a)^{n+1} = 0$ by the definition of k . By the extended Rolle's theorem, $g^{(n+1)}(\xi) = 0$ for some ξ in I . That means

$$\begin{aligned} 0 &= g^{(n+1)}(\xi) \\ &= f^{(n+1)}(\xi) - T_{n,a}^{(n+1)}(x) - k \frac{d^{n+1}}{dx^{n+1}}(x-a)^{n+1} \Big|_{x=\alpha} \\ &= f^{(n+1)}(\xi) - (n+1)! k \\ 0 &= f^{(n+1)}(\xi) - (n+1)! \frac{f(b) - T_{n,a}(b)}{(b-a)^{n+1}}. \end{aligned}$$

The equation in Taylor's theorem is just a rearrangement of this last one.

As an example, you can construct some Maclaurin polynomials for $f(x) = \sin x$ at $a = 0$. Note that

$$f(0) = 0 \quad f'(0) = 1 \quad f''(0) = 0 \quad f^{(3)}(0) = -1$$

and subsequent derivatives repeat these values cyclically. Each polynomial of odd degree n is identical to the next one, hence its error term has degree $n + 2$. Since the derivatives of the sine are positive or negative sines or cosines, you can easily find error bounds. For $n = 7$, for instance, here's the Maclaurin polynomial, its error term, and an error bound:

$$\begin{aligned} T_{7,0}(x) &= T_{8,0}(x) = x - x^3/3! + x^5/5! - x^7/7! \\ \frac{|f^{(9)}(\xi)|}{9!} |x|^9 &\leq \frac{|x|^9}{9!}. \end{aligned}$$

The following table shows the values of these bounds for $x = 3\pi/2$ and $n = 1$ to 13. The polynomial approximations to $\sin x = -1$ are also shown, as well as the actual error.

m	Error Bound	$T_{m,0}(x)$	Actual Error
1	1.7E+1	4.7124	5.7E+0
3	1.9E+1	-12.7286	-1.2E+1
5	1.0E+1	6.6367	7.6E+0
7	3.2E+0	-3.6023	-2.6E+0
9	6.4E-1	-0.4444	5.6E-1
11	9.1E-2	-1.0819	-8.2E-2
13	9.6E-3	-0.9911	8.8E-3

Figure 1 compares graphs of all but the last of these polynomials with that of the sine.

Because of the size of the errors, Maclaurin polynomials do not provide a reasonable way of calculating $\sin x$ except for very small values of x . Other methods are used in practice.

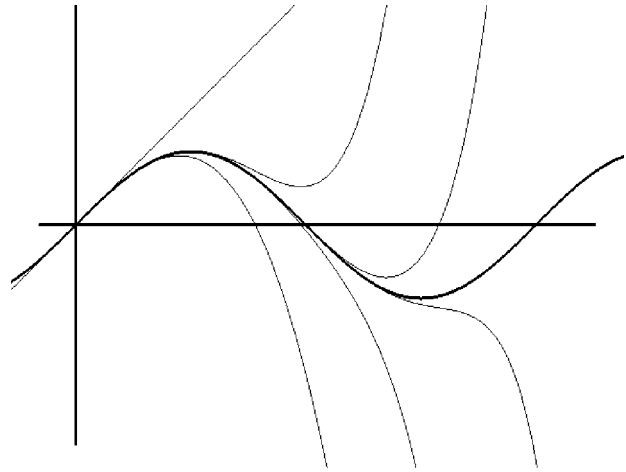


Figure 1 $\sin x$ and its first few Maclaurin polynomials