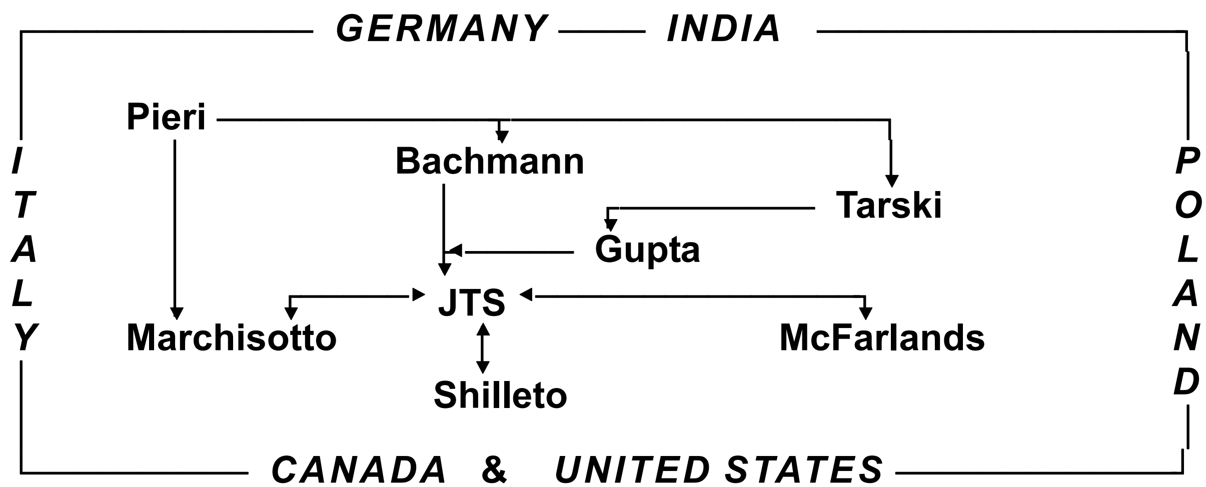
Magical Connections in Geometry
1900–2016



Colloquium, Department of Mathematics and Statistics
University of Regina

4 November 2016

Magical Connections in Geometry
Regina – Tarski – Pieri

❋❋❋ 6–7 April 1970 ❋❋❋

James T. Smith, PhD'70, Professor Emeritus
San Francisco State University

Years of Advanced Learning

- **JTS**

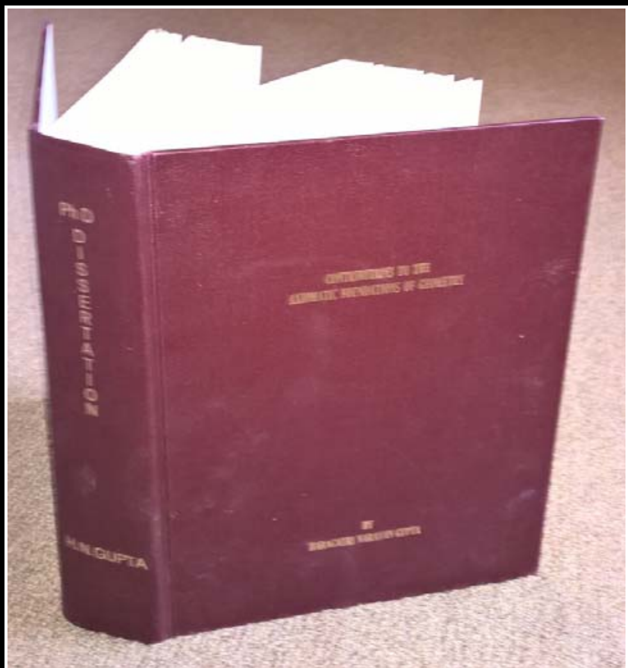
  1961–1967: 2 master’s, Navy work, SF State teaching

- **Haragauri N. Gupta**

  1945–1959: 2 master’s, teaching and admin in India
  Study in Germany

  1960–1966: Berkeley PhD (at age 40) from Alfred Tarski
  Stanford postdoc

H. N. Gupta
Contributions to the Axiomatic Foundations of Geometry



JTS took a course
on this in 1966.

Gupta & Rolando Chuaqui
near Hull, 1966

Regina

- Gupta ➜ Regina
  Fall 1966

- JTS ➜ Regina, Fall 1967
  - to study with Gupta
  - and watch on CBC
    the SF State riots!

Kiel, April 1969

- JTS researched not in the Tarski/Gupta line but in that of Friedrich Bachmann, in Kiel.

- Gupta, JTS ➜ Kiel, Oberwolfach

- July 1969:
  Gupta said “enough”!
- JTS ➜ SF State

Days of Magic

Regina
6–7 April 1970

UNIVERSITY OF SASKATCHEWAN, REGINA, CANADA

DEPARTMENT OF MATHEMATICS

Seminars in Geometry and Foundations

On the occasion of the final oral examination (defense of dissertation) for the first Ph.D. candidate (James T. Smith of San Francisc in the Department of Mathematics, we are fortunate to have present at the University two outstanding mathematicians in the field of foundations of geometry: Professors A. Tarski of Berkeley and F. Bachmann of Kiel. Both have accepted our invitations to present seminars of rather general interest to the mathematics community.

The following outline of formal activities have been scheduled for the period April 6 and 7 when our visitors will be on campus:

Monday, April 6, 1970 at 1:30 p.m. in C-100

Defense of Ph.D. dissertation entitled FOUNDATIONS OF METRIC GEOMETRY OF ARBITRARY DIMENSION by candidate James T. Smith of San Francisco. To this defense are invited any members of the Mathematics staff who care to attend. Professor Tarski is the official external examiner; Professor Bachmann will be present as one who has been closely associated with both Smith and his supervisor, Professor Gupta.

Monday, April 6, 1970, at 8:30 p.m. in C-22

Seminar Reflections and Absolute Geometry by Professor F. Bachmann of Kiel, Germany. See the attached Appendix A for details.

Tuesday, April 7, 1970 at 1:30 p.m. in C-25

Seminar entitled: Some Difficult Problems in Elementary Geometry by Professor A. Tarski of Berkeley. See attached Appendix B for details.

It is hoped that all members of the Mathematics staff, guests from neighboring universities, interested staff members in other department mathematics students and others interested in mathematics, particularly in geometry, will avail themselves of the opportunities to attend the lectures by two such distinguished visiting mathematicians.

C. L. Kaller
Chairman
Department of Mathematics

CLK/cm
2 appendices

Friedrich Bachmann
in 1969

Alfred Tarski
in 1968

At My Thesis Defense

- Age 30
- Tarski asked about the background of my research.
- I mentioned Bachmann’s 1959
  Aufbau der Geometrie aus dem Spiegelungsbegriff,
  the culmination of German work
  during 1900–1950.
- Tarski said I should study Mario Pieri’s 1900
  Monografia del punto e del moto.

Tarski's Lecture

Some Difficult Problems in Elementary Geometry

- **Familiar theorem:**

  If two polygons *V*, *W* have = areas, then for some *n* they can each be divided into *n* subpolygons, so that corresponding pairs are *congruent*.

- **See the example.**
- **What is the *smallest n* —their *degree of equivalence*?**
- **Tarski had studied this in general in his 1931–1932 paper *O stopniu równoważności wielokątów*, addressed to high-school teachers and gifted students.**

- **Solution for**

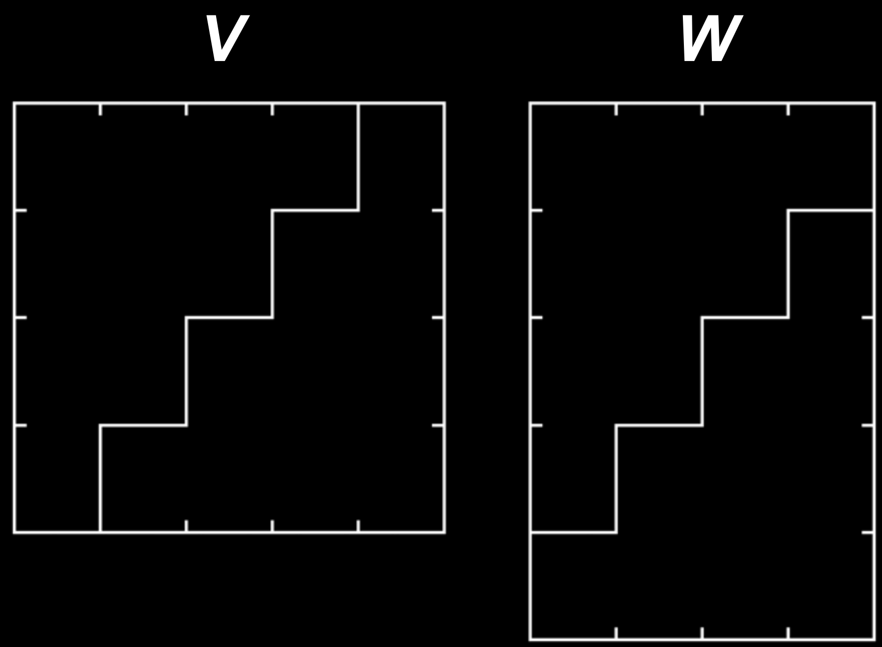
  V = unit square

  W = $^{(p+1)}/_{p}$

  - by -

  $^{p}/_{(p+1)}$

- **Degree of equivalence = 2**
- **By Henryk Moese, a teacher, in 1932.**



- **Moese conjectured, *This is the only way*.**
- **Adolf Lindenbaum, 1937, without proof: *Yes!***

Alfred Tarski

- 1901
  - Born Alfred Teitelbaum in Warsaw, to a
  - Jewish merchant family, in a culture of
  - prosperity and antisemitism,
  - under Russian oppression until 1915
- 1918: *Independent Poland*
  - Start and expansion of its university system
- 1920: Turmoil of the Polish-Soviet War
- 1924
  - PhD under Stanisław Leśniewski in logic
  - Changed name to Tarski
  - Banach–Tarski decomposition

- **1924→ Few open professorships**
  - **Full-time high-school teacher**
  - **Part-time lecturer: logic research, teacher prep**
  - **Studied foundations of the geometry he taught:**
  - **Pieri, *Elementary Geometry Based on the Notions of 'Point' and 'Sphere'* (1908)**

BIBLJOTEKA „WEKTORA". SERJA A. № 3.

MARIO PIERI.

GIEOMETRJA ELEMENTARNA

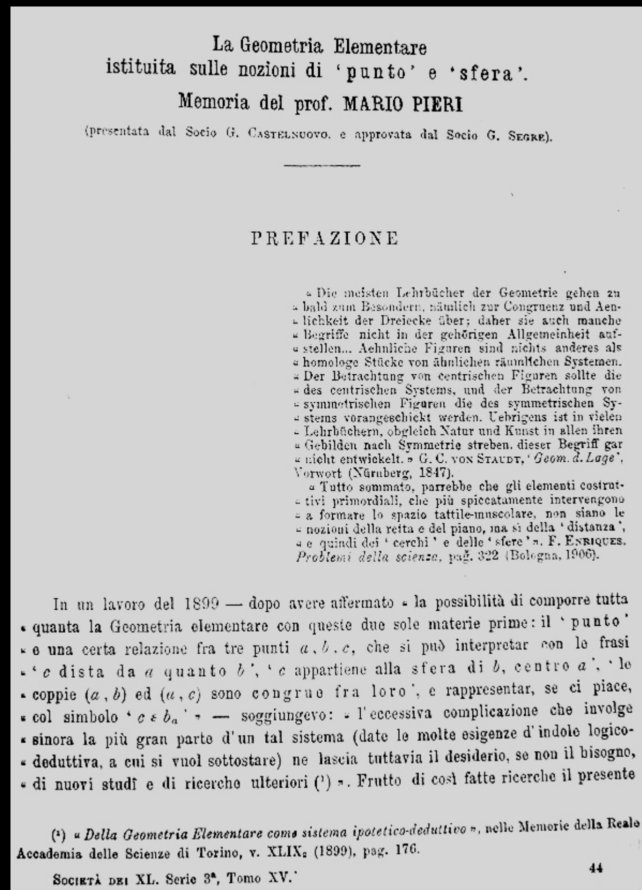
OPARTA NA POJĘCIACH „PUNKTU" I „KULI".

PRZEDSTAWIŁ WŁOSKIEMU TOWARZYSTWU NAUKOWEMU G. CASTELNUOVO, APROBOWAŁ G. SEGRE.

Z ORYGINAŁU WŁOSKIEGO PRZEŁOŻYŁ STEFAN KWIETNIEWSKI.

Z zapomogi Kasy pomocy dla osób pracujących na polu naukowem im. D-ra Józefa Mianowskiego.

WARSZAWA.
SKŁAD GŁÓWNY W KSIĘGARNI GEBETHNERA I WOLFFA.
1915.



La Geometria Elementare
istituita sulle nozioni di 'punto' e 'sfera'.
Memoria del prof. MARIO PIERI
(presentata dal Socio G. Castelnuovo, e approvata dal Socio G. Segre).

PREFAZIONE

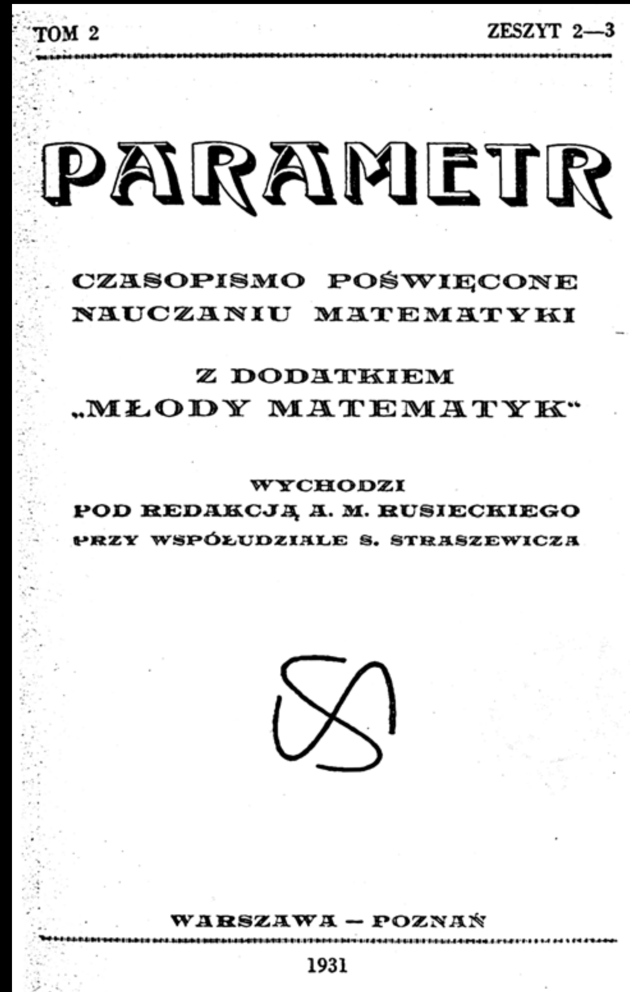
« Die meisten Lehrbücher der Geometrie gehen zu bald zum Besondern, nämlich zur Congruenz und Aehnlichkeit der Dreiecke über; daher sie auch manche Begriffe nicht in der gehörigen Allgemeinheit aufstellen... Aehnliche Figuren sind nichts anderes als homologe Stücke von ähnlichen räumlichen Systemen. Der Betrachtung von centrischen Figuren sollte die des centrischen Systems, und der Betrachtung von symmetrischen Figuren die des symmetrischen Systems vorangeschickt werden. Uebrigens ist in vielen Lehrbüchern, obgleich Natur und Kunst in allen ihren Gebilden nach Symmetrie streben, dieser Begriff gar nicht entwickelt. » G. C. von Staudt, '*Geom. d. Lage*', Vorwort (Nürnberg, 1847).

« Tutto sommato, parrebbe che gli elementi costruttivi primordiali, che più spiccatamente intervengono a formare lo spazio tattile-muscolare, non siano le nozioni della retta e del piano, ma sì della 'distanza', e quindi dei 'cerchi' e delle 'sfere' ». F. Enriques, *Problemi della scienza*, pag. 322 (Bologna, 1906).

In un lavoro del 1899 — dopo avere affermato « la possibilità di comporre tutta quanta la Geometria elementare con queste due sole materie prime: il 'punto' e una certa relazione fra tre punti a, b, c, che si può interpretar con le frasi 'c dista da a quanto b', 'c appartiene alla sfera di b, centro a', 'le coppie (a, b) ed (a, c) sono congrue fra loro', e rappresentar, se ci piace, col simbolo '$c \varepsilon b_a$' » — soggiungevo: « l'eccessiva complicazione che involge sinora la più gran parte d'un tal sistema (date le molte esigenze d'indole logico-deduttiva, a cui si vuol sottostare) ne lascia tuttavia il desiderio, se non il bisogno, di nuovi studî e di ricerche ulteriori (¹) ». Frutto di così fatte ricerche il presente

(¹) « *Della Geometria Elementare come sistema ipotetico-deduttivo* », nelle Memorie della Reale Accademia delle Scienze di Torino, v. XLIX₂ (1899), pag. 176.

Tarski in the 1930s



TOM 2 ZESZYT 2—3

PARAMETR

CZASOPISMO POŚWIĘCONE NAUCZANIU MATEMATYKI

Z DODATKIEM „MŁODY MATEMATYK"

WYCHODZI POD REDAKCJĄ A. M. RUSIECKIEGO PRZY WSPÓŁUDZIALE S. STRASZEWICZA

WARSZAWA — POZNAŃ

1931

- Renowned logic researcher
- Teacher & teacher educator
- His problem appeared in *The Young Mathematician*:

ROK 1 MARZEC 1931 NR. 3

MŁODY MATEMATYK

CZASOPISMO DLA MŁODZIEŻY SZKOLNEJ WYCHODZI POD REDAKCJĄ A. M. RUSIECKIEGO PRZY WSPÓŁUDZIALE S. STRASZEWICZA.

TREŚĆ: J. Gadomski. Odkrycie Plutona. — A. Tarski. O stopniu równoważności wielokątów. — Rozwiązania zadań Nr. 40 i 57. — Zadania Nr. 167 — 181.

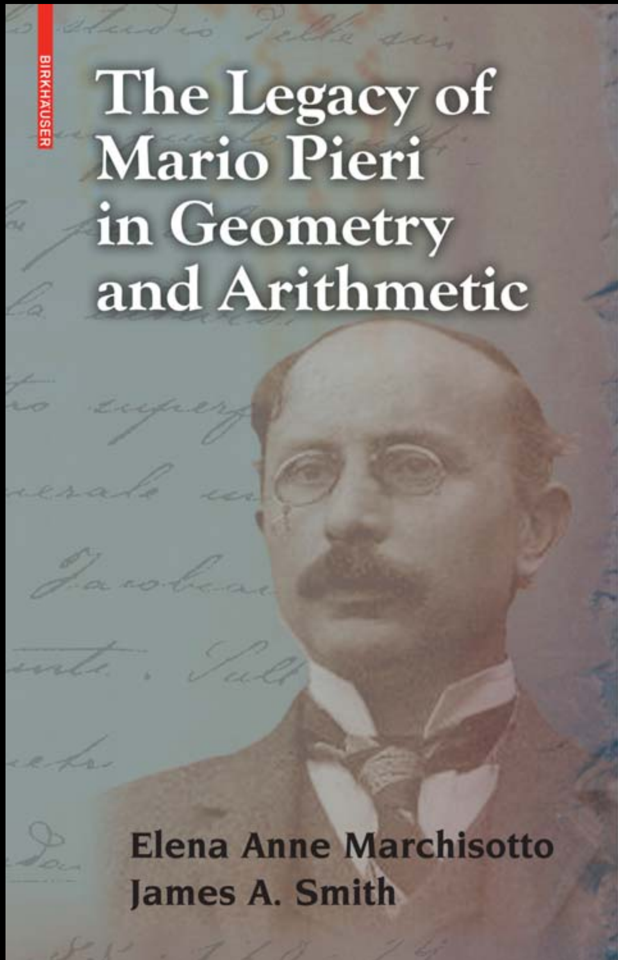
JTS

- 1970–1983 Teaching & some research à la Bachmann
- 1975–1982 Much administration
- 1984–2003 Teaching & much software & publishing
- 2003➜ *Time to retire to something new*

- Elena A. Marchisotto sought a collaborator for a book on Pieri.

- I responded & learned to translate Italian.

- A book and an award-winning *Monthly* paper resulted in 2007, 2010.



Elementary Geometry Based on the Notions of Point and Sphere

Memoir by Prof. MARIO PIERI

presented by Member G. CASTELNUOVO and approved by Member C. SEGRE[4]

PREFACE

> "Most textbooks on geometry proceed too quickly to the particular, that is to congruence and similarity of triangles; therefore they also fail to present many concepts in appropriate generality. Similar figures are nothing more than homologous pieces of similar[5] spatial systems. Consideration of centered figures and symmetric figures [in general] should begin with consideration of centered systems and symmetric systems. Moreover, although nature and art strive for symmetry in all their forms, in many textbooks this concept is not developed." G. C. VON STAUDT 1847, Foreword.

> "In summation it would seem that the primary constructive elements, which more evidently combine to form tactile-physical space, are not the notions of line and plane, but of 'distance,' and therefore of 'circles' and 'spheres.' " F. ENRIQUES 1906, page 322.[6]

In an 1899 work,[7] I affirmed "the possibility of composing all elementary geometry from just two prime materials: 'point' and a certain relation among three points a,b,c that can be rendered by the phrases 'c is as distant from a as b is,' 'c belongs to the sphere through b with center a,' or 'the pairs a,b and a,c are congruent,' and that can be represented, if you please, by the formula $c \in b_a$." Then I remarked, "the excessive

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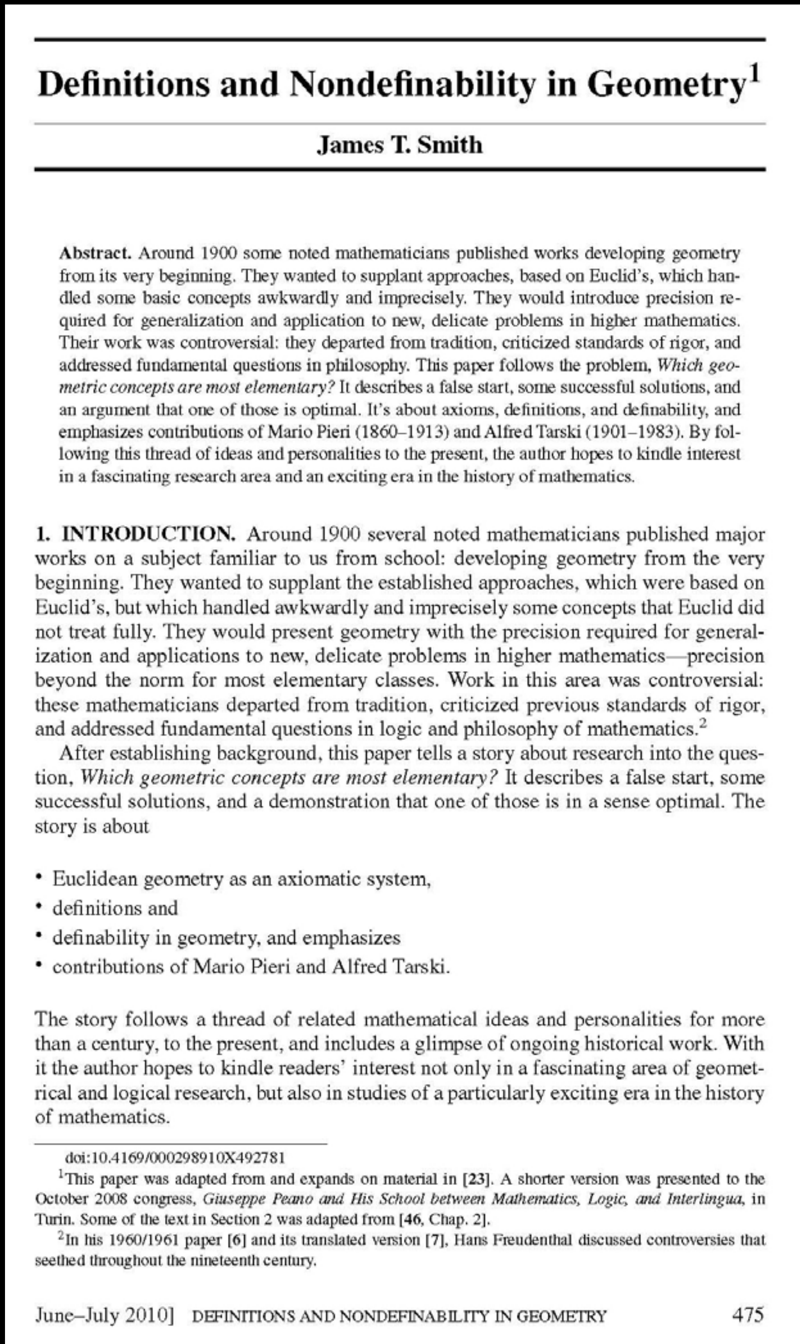
[4] [The original named G. SEGRE, evidently in error.]

[5] [Translated from the German quotation in Pieri's paper. Pieri altered Staudt's text, de-emphasizing the reduction of three-dimensional concepts to plane concepts. The sense of Staudt's text can be restored by (1) inserting "plane systems, and similar solids, homologous pieces of similar" immediately after this footnote reference and (2) inserting "plane" immediately before the first occurrence of "figures" and the latter two occurrences of "systems."]

[6] [The quotation also occurs in the [1910] 1985 second edition, 186–187. Enriques tried to distinguish the most basic geometric concepts by considering psychological and physiological aspects of their perception. In particular, he considered visual, tactile, and muscular aspects. Tactile senses include pressure, which is opposite to muscular exertion. Pieri changed Enriques's words slightly, but the only significant difference is at the beginning. Enriques wrote, "A muscular definition of 'line' is also possible, based on its mechanical properties, but in comparison, it seems that ...," then continued as in Pieri's version.]

[7] [Pieri 1900a.]



Definitions and Nondefinability in Geometry[1]

James T. Smith

Abstract. Around 1900 some noted mathematicians published works developing geometry from its very beginning. They wanted to supplant approaches, based on Euclid's, which handled some basic concepts awkwardly and imprecisely. They would introduce precision required for generalization and application to new, delicate problems in higher mathematics. Their work was controversial: they departed from tradition, criticized standards of rigor, and addressed fundamental questions in philosophy. This paper follows the problem, *Which geometric concepts are most elementary?* It describes a false start, some successful solutions, and an argument that one of those is optimal. It's about axioms, definitions, and definability, and emphasizes contributions of Mario Pieri (1860–1913) and Alfred Tarski (1901–1983). By following this thread of ideas and personalities to the present, the author hopes to kindle interest in a fascinating research area and an exciting era in the history of mathematics.

1. INTRODUCTION. Around 1900 several noted mathematicians published major works on a subject familiar to us from school: developing geometry from the very beginning. They wanted to supplant the established approaches, which were based on Euclid's, but which handled awkwardly and imprecisely some concepts that Euclid did not treat fully. They would present geometry with the precision required for generalization and applications to new, delicate problems in higher mathematics—precision beyond the norm for most elementary classes. Work in this area was controversial: these mathematicians departed from tradition, criticized previous standards of rigor, and addressed fundamental questions in logic and philosophy of mathematics.[2]

After establishing background, this paper tells a story about research into the question, *Which geometric concepts are most elementary?* It describes a false start, some successful solutions, and a demonstration that one of those is in a sense optimal. The story is about

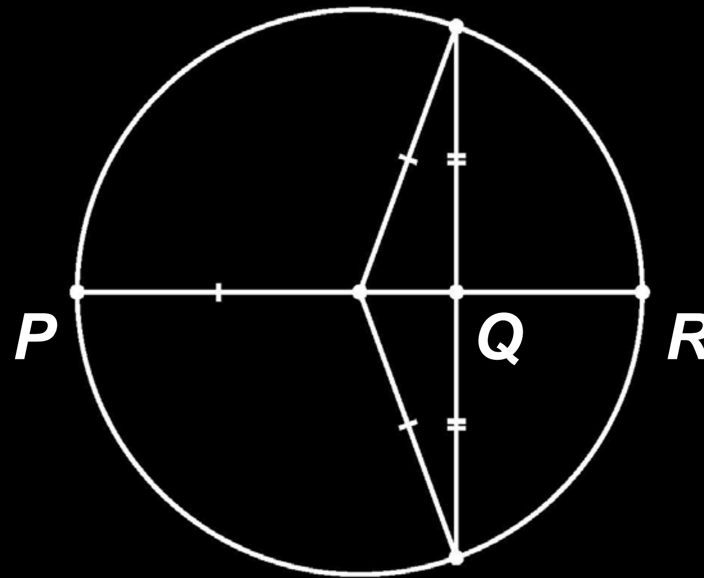
- Euclidean geometry as an axiomatic system,
- definitions and
- definability in geometry, and emphasizes
- contributions of Mario Pieri and Alfred Tarski.

The story follows a thread of related mathematical ideas and personalities for more than a century, to the present, and includes a glimpse of ongoing historical work. With it the author hopes to kindle readers' interest not only in a fascinating area of geometrical and logical research, but also in studies of a particularly exciting era in the history of mathematics.

doi:10.4169/000298910X492781

[1]This paper was adapted from and expands on material in [23]. A shorter version was presented to the October 2008 congress, *Giuseppe Peano and His School between Mathematics, Logic, and Interlingua*, in Turin. Some of the text in Section 2 was adapted from [46, Chap. 2].

[2]In his 1960/1961 paper [6] and its translated version [7], Hans Freudenthal discussed controversies that seethed throughout the nineteenth century.



- **1908: Pieri defined betweenness *P-Q-R* in terms of equidistance.**
- **1935: Tarski showed equidistance is *not* definable in terms of betweenness.**

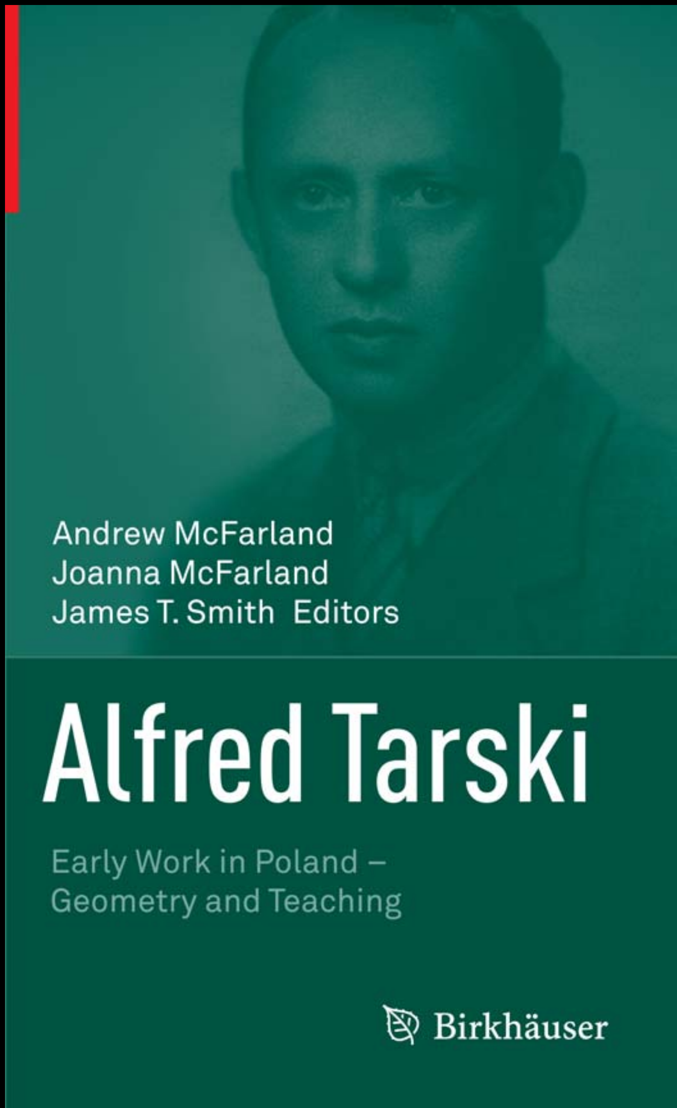
State of Jefferson, 2007

- Presenting this paper, I said that some neat papers of Tarski still needed translation from Polish.

- Andrew and Joanna McFarland responded.

- We collaborated, and the project turned into a major book in 2014.

Interviewing Witold Kozłowski, who was Tarski’s high-school student during 1934-1938.



7.2 *The Degree of Equivalence of Polygons* (1931)

This section contains an English translation of Alfred Tarski's paper *O stopniu równoważności wielokątów*, [1931] 2014a, which appeared in volume 1 of the journal *Młody matematyk*. Aimed at particularly interested gimnazjum students, this journal was distributed with another one, *Parametr*, addressed to their teachers.[20] A draft English translation by Izaak Wirszup was published informally by the University of Chicago as Tarski and Moese 1952. The present translation was carried out independently, but used Wirszup's for confirmation.

This translation is meant to be as faithful as possible to Tarski's original text. Its only intentional modernizations are punctuation, and some changes in symbols, where Tarski's conflict with others used throughout this book. A bibliographic reference has been adjusted to conform with the conventions of the present book. All [square] brackets in the translation enclose editorial comments inserted for clarification, often as footnotes. As an aspect of adjusting punctuation, the editors increased white space to enhance visual organization.

DR. ALFRED TARSKI (Warsaw)

On the Degree of Equivalence of Polygons

In this article I want to discuss some concepts, belonging entirely to the realm of elementary geometry, which until now have been investigated hardly at all.

As is well known, we call two polygons W and V *equivalent*, expressing this with the formula $W \equiv V$, if they can be divided into the same number of respectively congruent polygons.[21] This subdivision of equivalent polygons into congruent parts is not unique: two equivalent polygons can be divided into congruent parts in various ways, with respect to the number as well as the form of these parts.

We explain this with an example: figure 1, and figure 2 as well, show that a square with edge a and a rectangle with edges $\frac{5}{4}a$ and $\frac{4}{5}a$ are equivalent to each other, but their subdivisions in the two figures are quite distinct.

In connection with this observation, a question arises in a natural way: what is the *least* number of respectively congruent parts into which two given equivalent polygons can be divided? We want to touch upon a problem of exactly this type in *Parametr*.

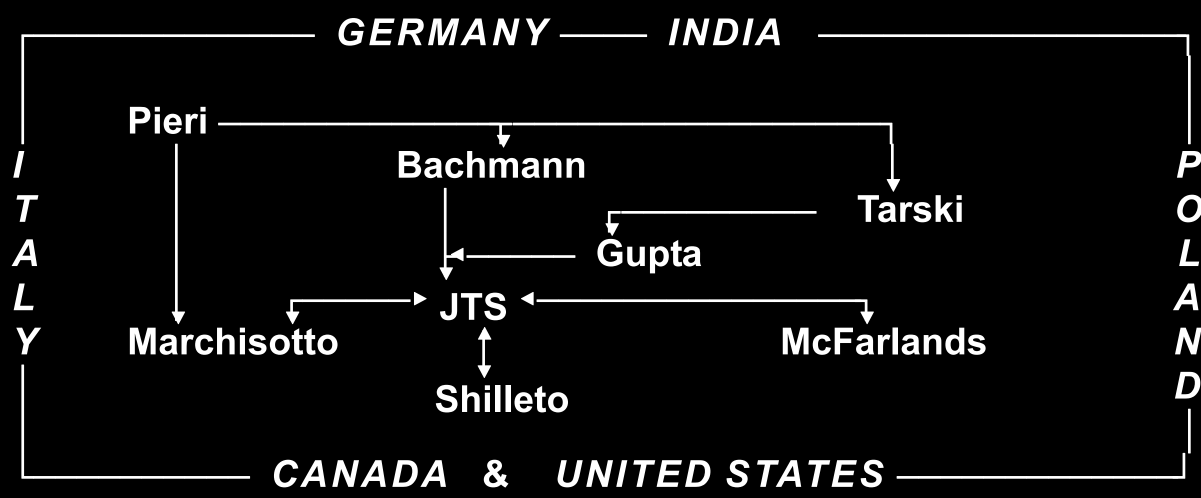
[20] For further information about these journals, consult sections 7.1 and 9.7.

[21] [This notion of equivalence is discussed in section 4.1. It is different from that used in chapters 5 and 6. The two notions are compared succinctly on page 59.]

Current Work

- Dr. James R. Shilleto attended Tarski's 1970 Regina talk.
- He sketched a proof that the "staircase" is the *only* solution to Tarski's problem.
- We're drafting a joint paper about that proof.

Magical Connections in Geometry
1900–2016



Thank you for your interest,

and for helping me start a wonderful career!

James T. Smith, PhD ’70

Professor Emeritus
San Francisco State University