INITIAL VALUE PROBLEMS FOR SYSTEMS OF FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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To pose an *initial value problem for a system of* m *first-order ordinary differential equations*, specify

- a nonempty closed interval $I = [x_0, x_0 + H]$,
- a vector $U^{(0)} \in \mathbb{R}^m$ of initial values, and
- a function $F(x, Y)$ defined for all $x \in I$ and $Y \in \mathbb{R}^m$, whose values lie in $\mathbb{R}^m$.

A *solution* of the problem is a differentiable function $U : I \to \mathbb{R}^m$ such that

- $U(x_0) = U^{(0)}$, and
- $U'(x) = F(x, U(x))$ for all $x \in I$.

(The derivative U' of a vector valued function U is the vector valued function whose components are the derivatives of U.) You can easily generalize the discussion of ordinary differential equation initial value problems for the familiar case $m = 1$ to apply to any m.

To pose an *initial value problem for an mth-order ordinary differential equation*, specify

- a nonempty closed interval $I = [x_0, x_0 + H]$,
- a vector $U^{(0)} \in \mathbb{R}^m$ of initial values, and
- a function $f(x, Y)$ defined for all $x \in I$ and $Y \in \mathbb{R}^m$.

A *solution* of the problem is an m times differentiable function u on I such that

$$\begin{aligned} u(x_0) &= u^{(0)}{}_0 \\ u'(x_0) &= u^{(0)}{}_1 \\ &\vdots \\ &\vdots \\ u^{(m-1)}(x_0) &= u^{(0)}{}_{m-1} \end{aligned}$$

(In this context it's convenient to start subscripts at 0.)

$$u^{(m)}(x) = f(x, u(x), u'(x), \ldots, u^{(m-1)}(x)) \qquad \text{for all } x \in I.$$

7 December 2002

For example, consider the problem

$$m = 2 \qquad U^{(0)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad f(x,Y) = -y_0 \qquad Y = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$
$$x_0 = 2$$

A solution is a twice differentiable function u on $I = [0,H]$ such that

$$\begin{aligned} u(0) &= u^{(0)}{}_0 = 0 \\ u'(0) &= u^{(0)}{}_1 = 1 \\ u''(x) &= f(x,u(x),u'(x)) = -u(x). \end{aligned}$$

The familiar solution is $u(x) = \sin x$: it's unique.

You can convert any mth order initial value problem to an equivalent initial value problem for a system of m first order equations: given the interval I, the vector $U^{(0)}$ of initial values, and the function $f(x,Y)$ for the mth order problem, set up the function

$$F(x,Y) = \begin{bmatrix} y_1 \\ \vdots \\ y_{m-1} \\ f(x,Y) \end{bmatrix} \qquad Y = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{m-1} \end{bmatrix}$$

for the system problem. When $m = 2$, for instance, this amounts to converting the problem of finding a twice differentiable function u such that

$$u(x_0) = u^{(0)}{}_0 \qquad u''(x) = f(x,u(x),u'(x))$$
$$u'(x_0) = u^{(0)}{}_1$$

to the equivalent one of finding differentiable functions u_0 and u_1 such that

$$u_0(x_0) = u^{(0)}{}_0 \qquad u_0{}'(x) = u_1(x)$$
$$u_1(x_0) = u^{(0)}{}_1 \qquad u_1{}'(x) = f(x,u_0(x),u_1(x)).$$

For the earlier example, the second order equation for the sine, the construction looks like this:

Convert	To	
$u(0) = 0$	$u_0(0) = 0$	$u_0' = u_1$
$u'(0) = 1$	$u_1(0) = 1$	$u_1' = -u_0$
$u'' = -u$		

Clearly, you can use this device to convert an initial value problem for a system of ordinary differential equations of various orders into an equivalent problem for a first order system. Since the theory and the algorithms generalize so readily from single first order equations to first order systems, you can restrict the formal discussion to first order equations. That simplifies the notation.