

FUNDAMENTAL THEOREM OF ALGEBRA

James T. Smith
San Francisco State University

According to Gauss's *fundamental theorem of algebra*, every nonconstant polynomial p with complex coefficients has a complex root. You meet this result first in precalculus mathematics, then find that *many* studies in pure and applied mathematics are based on it. Its proof is generally postponed until graduate courses. The first proof I learned follows from Cauchy's integral theorem in complex analysis. It's elegant, but it uses a wide range of facts about continuity and differentiability of complex functions *and* the complicated apparatus of contour integrals. For years I hunted for a proof that's simple enough to present as a sidebar in undergraduate courses that *use* the fundamental theorem. I think I found one, in Remmert 1993. That author attributes the following proof to J. Argand, 1814.

By Cauchy's bound, all roots of a polynomial $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ lie in the closed disk D centered at 0 with radius

$$1 + \max_{k=0}^n \left| \frac{a_k}{a_n} \right|.$$

Since polynomials and the absolute value function are continuous, and any composition of continuous functions is continuous, the real-valued function $q : z \rightarrow |p(z)|$ on D is continuous, hence it assumes a minimum value. That is, there exists $c \in D$ such that $q(c) \leq q(w)$ for all $w \in D$. (These properties of continuous functions weren't stated and proved until about fifty years after Argand's work; Remmert says that Argand would have thought them obvious, not needing proof.)

Argand showed $p(c) = 0$ by assuming the opposite and deriving a contradiction. Under that opposite assumption the function $h : z \rightarrow p(c)^{-1}p(z + c)$ would be a nonconstant polynomial. That is, there would exist integers k and n such that $1 \leq k \leq n$, and complex numbers b_k, \dots, b_n such that $b_k \neq 0$ and $h(z) = 1 + b_kz^k + \cdots + b_nz^n$ for all complex z . Find d such that $d^k = -1/b_k$. The polynomial $g : z \rightarrow b_{k+1}d^{k+1}z + \cdots + b_nd^n z^{n-k}$ is continuous and $g(0) = 0$, so there exists $\delta > 0$ such that $|g(t)| < 1/2$ for all t such that $0 \leq t < \delta$. Select any t such that $0 < t < 1, \delta$ and let $w = dt + c$. Then

$$\begin{aligned} |h(dt)| &= |1 + b_k(dt)^k + \cdots + b_n(dt)^n| \\ &= |1 + b_kd^k t^k + b_{k+1}d^{k+1}t^{k+1} + \cdots + b_nd^n t^n| \\ &= |1 - t^k + t^k g(t)| \leq |1 - t^k| + |t^k g(t)| \\ &= 1 - t^k + t^k |g(t)| < 1 - t^k + 1/2 t^k < 1 - 1/2 t^k < 1 \end{aligned}$$

$$q(w) = |p(dt + c)| = |p(c)h(dt)| = |p(c)||h(dt)| < |p(c)| = q(c).$$

This would contradict the previous paragraph, which established $q(c) \leq q(w)$.

Reference

Remmert, Reinhold. 1993. Vom Fundamentalsatz der Algebra zum Satz von Gelfand-Mazur. *Mathematische Semesterberichte* 40: 63–71.