

EQUIVALENCES AND PARTITIONS

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An *equivalence* on a set X is a binary relation E on X that's *reflexive*, *symmetric*, and *transitive*: for all $x, y, z \in X$,

$$\begin{aligned}x & E x \\x & E y \Rightarrow y E x \\x & E y \ \& \ y E z \Rightarrow x E z .\end{aligned}$$

This is the mathematical explication of the informal concept *sameness*. Clearly, the identity I_X is the smallest equivalence on X , and $X \times X$ is the largest.

Let E be an equivalence on X . Each $x \in X$ belongs to its *equivalence class*

$$x/E = \{ y \in X : x E y \} .$$

Two equivalence classes x/E and y/E are equal just when $x E y$. The function $x \mapsto x/E$ is called the *quotient map*. Its range is the family X/E of all equivalence classes.

Suppose $f : X \rightarrow Y$ surjectively. Define a binary relation E_f on X by setting

$$x E_f y \Leftrightarrow f(x) = f(y)$$

for all $x, y \in X$. Then E_f is an equivalence on X . Moreover, you can construct a bijection $g : X/E_f \rightarrow Y$ such that for each $x \in X$, $g(x/E_f) = f(x)$. Proceed as follows. First, show there's a unique $y \in Y$ —namely $y = f(x)$ —such that $f(x') = y$. for all $x' \in x/E_f$. Then define $g(x/E_f) = y$.

$$\begin{array}{ccc} & f & \\ X & \rightarrow & Y \\ \downarrow & \nearrow g & \\ X/E_f & & \end{array}$$

In the previous paragraph, if $Y = X/E$ for some equivalence E on X , and $f : X \rightarrow Y$ is the quotient map, then $E_f = E$ and g is the identity I_Y .

A *partition* of X is a family \mathcal{F} of disjoint subsets of X whose union is X . This is the mathematical explication of the informal concept *classification*. You've an example already: if E is an equivalence on X , then X/E is a partition of X . The partition corresponding to the equivalence I_X consists of all singletons of members of X ; it's the *finest* partition. The *coarsest* partition corresponds to the equivalence $X \times X$: it consists of the set X alone.

Suppose \mathcal{F} is a partition of X . Define a binary relation $E_{\mathcal{F}}$ on X by setting

$$x E_{\mathcal{F}} y \Leftrightarrow \exists A [A \in \mathcal{F} \ \& \ x, y \in A]$$

for all $x, y \in X$. Then $E_{\mathcal{F}}$ is an equivalence on X , and $X/E_{\mathcal{F}} = \mathcal{F}$.

In the previous paragraph, if $\mathcal{F} = X/E$ for some equivalence E on X , then $E_{\mathcal{F}} = E$.

These results show that the notions *equivalence* and *partition* are merely two aspects of the same concept.

Routine Exercises

1. Show that a relation E on X is
 - a) reflexive if and only if $I_X \subseteq E$;
 - b) symmetric if and only if $E \subseteq \check{E}$;
 - c) transitive if and only if $E \mid E \subseteq E$.

Show that for an equivalence E on X , actually $E = \check{E}$ and $E \mid E = E$.

2. Define binary relations E and F on the set X of Americans by setting for each $x, y \in X$,

$x E y$ if and only if x and y have the same age, in years;
 $x F y$ if and only if x and y differ in age by less than one year.

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Are E and F equivalences on X ?

- Find a relation on the set $\{0, 1\}$ that's nonempty, symmetric, and transitive, but not reflexive.
- Show that the intersection of a nonempty family of equivalences on X is an equivalence on X . Let \mathcal{E} be a nonempty family of equivalences on X such that for any $D, E \in \mathcal{E}$ there exists $F \in \mathcal{E}$ with $D, E \subseteq F$. Show that $\bigcup \mathcal{E}$ is an equivalence on X .
- Let \mathcal{I} and \mathcal{I}' be partitions of X . \mathcal{I} is called *finer* than \mathcal{I}' if each member of \mathcal{I} is included in some member of \mathcal{I}' . Show that this is true just when $E \subseteq E'$, where E and E' are the equivalences corresponding to \mathcal{I} and \mathcal{I}' .

Substantial problems

- Let R be a relation on X . Define $R^1 = R$ and for each integer $n > 1$,

$$R^n = R \mid \dots \mid R \quad (n \text{ copies of } R).$$

Define $R^* = \bigcup_n R_n$, the *transitive closure* of R . Prove that R^* is the smallest transitive relation on X that includes R .

- Let \check{R} be a relation such that $R \mid \check{R} \mid R \subseteq R$ and set $X = \text{Dom } R$. Prove that $R \mid \check{R}$ is an equivalence on X .

- Let $H(n)$ denote the number of partitions of an n element set. Show that

$$H(n + 1) = \sum_{m=0}^n \binom{n}{m} H(m).$$

Compute $H(10)$. Prove that for all real x ,

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$$e^{e^x-1} = \sum_{n=0}^{\infty} \frac{H(n)}{n!} x^n .$$

This result is due to G. Williams (1945).

4. Let E and F be equivalences on X and $E \mid F = F \mid E$. Prove that $E \mid F$ is the smallest equivalence on X that includes E and F .
5. For each integer $n > 0$, define a binary relation E_n on \mathbb{Z} by setting, for all integers a and b , $a E_n b$ if and only if n divides $a - b$. Show that E_n is an equivalence on \mathbb{Z} . Now consider two integers m and $n > 0$. Show that $E_m \subseteq E_n$ if and only if n divides m . Let p and q be the least common multiple and greatest common divisor of m and n . Show that $E_m \cap E_n = E_p$ and $E_m \mid E_n = E_n \mid E_m = E_q$.

Reference

A. G. Kurosh, *Lectures on general algebra*, Chelsea, 1963, pp. 18-20.