Definitions and Nondefinability in Geometry[1]

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Abstract. Around 1900 some noted mathematicians published works developing geometry from its very beginning. They wanted to supplant approaches, based on Euclid's, which handled some basic concepts awkwardly and imprecisely. They would introduce precision required for generalization and application to new, delicate problems in higher mathematics. Their work was controversial: they departed from tradition, criticized standards of rigor, and addressed fundamental questions in philosophy. This paper follows the problem, *Which geometric concepts are most elementary?* It describes a false start, some successful solutions, and an argument that one of those is optimal. It's about axioms, definitions, and definability, and emphasizes contributions of Mario Pieri (1860–1913) and Alfred Tarski (1901–1983). By following this thread of ideas and personalities to the present, the author hopes to kindle interest in a fascinating research area and an exciting era in the history of mathematics.

1. INTRODUCTION. Around 1900 several noted mathematicians published major works on a subject familiar to us from school: developing geometry from the very beginning. They wanted to supplant the established approaches, which were based on Euclid's, but which handled awkwardly and imprecisely some concepts that Euclid did not treat fully. They would present geometry with the precision required for generalization and applications to new, delicate problems in higher mathematics—precision beyond the norm for most elementary classes. Work in this area was controversial: these mathematicians departed from tradition, criticized previous standards of rigor, and addressed fundamental questions in logic and philosophy of mathematics.[2]

After establishing background, this paper tells a story about research into the question, *Which geometric concepts are most elementary?* It describes a false start, some successful solutions, and a demonstration that one of those is in a sense optimal. The story is about

- Euclidean geometry as an axiomatic system,
- definitions and
- definability in geometry, and emphasizes
- contributions of Mario Pieri and Alfred Tarski.

The story follows a thread of related mathematical ideas and personalities for more than a century, to the present, and includes a glimpse of ongoing historical work. With it the author hopes to kindle readers' interest not only in a fascinating area of geometrical and logical research, but also in studies of a particularly exciting era in the history of mathematics.

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[1]This paper was adapted from and expands on material in [**23**]. A shorter version was presented to the October 2008 congress, *Giuseppe Peano and His School between Mathematics, Logic, and Interlingua*, in Turin. Some of the text in Section 2 was adapted from [**46**, Chap. 2].

[2]In his 1960/1961 paper [**6**] and its translated version [**7**], Hans Freudenthal discussed controversies that seethed throughout the nineteenth century.

2. THE AXIOMATIC METHOD. For centuries, mathematicians agreed that geometry should be an applied science, the study of certain aspects of physical space. They regarded as valid those propositions commonly believed to follow from Euclid's postulates and definitions: the theorems of Euclidean geometry. This was based on their deep trust in the deductions, and the strong agreement between these theorems and observable phenomena that they described. Euclidean geometry was so successful in explaining and predicting properties of the physical world that for two millenia no alternative system was considered. Euclidean geometry was considered *correct*, not simply accurate and useful. A geometry whose theorems contradicted Euclid's would be regarded as incorrect and therefore useless. Analyzing Euclid's use of the parallel postulate and related issues, mathematicians eventually began to consider alternatives that did contradict that postulate. This generated controversy, as mentioned earlier, and scientists wrestled with the idea of a geometric postulate system as a correct description of physical space. Late in the nineteenth century, to facilitate that study, they began to reexamine the *axiomatic method* that Euclid used to construct his system.

The method's most memorable aspect is its *derivation* of theorems from others proved earlier, which were derived from others yet more basic, which were. . . , and so on. The most basic principles, necessarily left unproved, are called *axioms*. This deductive organization was described by Aristotle [**1**] around 350 B.C.E. Euclid famously but imperfectly employed it in his *Elements* [**3**] around 300 B.C.E. Filling the gaps required investigation of steps that Euclid and his successors for two millennia evidently regarded as not needing proof. In some cases that amounted to formulating axioms that they assumed but did not state. In particular, Euclid's reasoning about the sides of lines in a plane left so many gaps that one can imitate him closely and derive nonsense.[3]

The theorems of an axiomatic theory are about concepts, some *defined* in terms of others considered earlier, which were defined in terms of others yet more basic, which were. . . , and so on. The most basic concepts, necessarily left undefined, are called *primitive*. Aristotle also described this organization. But Euclid didn't employ it fully. He did define precisely most of the concepts he used, in terms of a few he introduced at the beginnings of his presentations. However, his "definitions" of those most basic concepts were not definitions in our sense, but vague, circular discussions. Nevertheless, Euclid's work gained such prestige that Aristotle's definitional technique was rarely mentioned again until the late 1800s.

At that time the relationship of various geometric theories was becoming involved and confusing. Mathematicians studied hyperbolic non-Euclidean geometry as deeply as Euclidean, and showed how to extend each to a projective geometry by introducing "ideal points at infinity." They could establish the use of homogeneous coordinates in the projective geometry and employ them with a polar system to measure distance. (To understand this story, readers need not know details of projective geometry—but only that sophisticated mathematical devices were used for this purpose.) Hyperbolic or Euclidean geometry could be reconstructed from the projective by using different polar systems. Felix Klein provided a comparative framework with his 1872 *Erlanger program* [**18**], which regarded projective geometry as the most basic. He also emphasized the study of the *automorphisms* of a geometry: the transformations that preserve its basic notions.

That complexity and the new projective methods used in algebraic geometry, the study of solutions of systems of multivariate algebraic equations, led to a new emphasis on the axiomatic method, in Aristotle's original form.

[3] David Hilbert noted this in 1897/1898: see [**53**, Sec. 4.2]. For an English discussion, see [**46**, Sec. 2.2].

3. PASCH. Moritz Pasch (1843–1930) was born to a merchant family in Breslau, Prussia (now Wrocław, Poland). He studied at the university there, earned the doctorate in 1865, and pursued an academic career in Giessen, Germany. Pasch wrote research papers and some influential texts, and served with great distinction as an administrator. His early research was in algebraic geometry, but he soon turned to foundations of analysis and geometry. He saw the need to repair faults in classical Euclidean geometry, particularly in reasoning about the sides of a line in a plane, and to firmly ground the powerful projective methods presented in 1847 by G. K. C. von Staudt [**47**]. To this end, Pasch published in 1882 the first completely rigorous axiomatic presentation of a geometric theory.[4]

Pasch noted that, in contrast to earlier practice, he would discuss certain concepts without definition. Determining which he actually left undefined requires close reading. They are

- *point*,
- segment *between* two points,
- *coplanarity* of a point set,
- *congruence* of point sets.

He defined all other geometric notions from those. For example, Pasch called three points *collinear* if they are not distinct or one lies between the other two, and defined the *line* determined by two distinct points to be the set of points collinear with them. Those definitions rely on logic: the words *not*, *distinct*, *or*, and *set* appear there. Arithmetic is not really required: *one*, *two*, and *three* could be avoided through use of variables and more logic.

As example axioms, consider a simple one, that there exist three noncoplanar points, and the famous, more complicated, *Pasch axiom* illustrated by Figure 1. These involve even more logic: *if*, *any*, and *exist*. The Pasch axiom directly implies transitivity of the relation satisfied by points A and B when they lie *on the same side* of a given line DF. (Let this mean that no point on DF should lie between A and B. If A and B lay on different sides of DF, then by the Pasch axiom, either A and C, or else B and C, would lie on different sides.)

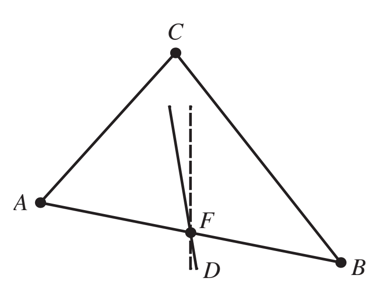

Figure 1. *The Pasch axiom*: If A, B, and C are distinct coplanar points and F is a point between A and B, then there exists on any line DF some point of segment AC or of BC.

Pasch developed the geometry of incidence, betweenness, and congruence, extended that to projective space, and then showed how to select a polar system to

[4]Heinrich Schröter supervised Pasch's doctoral study. In his autobiography [**32**, pp. 9–12], Pasch wrote that he could not continue working in higher mathematics until he settled questions about its foundations. The traditional presentations provided him no support, so he wrote his 1882 books [**29**] and [**30**] on foundations of analysis and geometry.

develop Euclidean or hyperbolic geometry. In spite of the abstract nature of his presentation, Pasch clearly indicated, as he introduced his axioms, that he regarded his system as a description of the physical world:

> In contrast to the propositions justified by proofs... there remains a group of propositions from which all others follow... based directly on observations....[5]

4. PEANO. The next character in this story stemmed from a very different milieu: a small farm near Cuneo in the Piedmont region of the Kingdom of Sardinia. Giuseppe Peano (1858–1932) was schooled in Cuneo and in the regional capital, Turin. By then this region was part of a unified Italy. Peano earned the doctorate in 1880 at the University of Turin and spent his career there and at the adjacent military academy. As a junior professor Peano undertook to reformulate with utmost precision all of pure mathematics! His 1889 booklet [**33**] on foundations of geometry contained some technical improvements over Moritz Pasch's book [**30**].[6] More importantly, Peano departed from Pasch's then prevalent approach by divorcing that discipline from the study of the physical world:

> Depending on the significance attributed to the undefined symbols... the axioms can be satisfied or not. If a certain group of axioms is verified, then all the propositions that are deduced from them will be equally true... [**33**, §1, note].

This freedom to consider various interpretations of the undefined, or primitive, concepts, and the distinction between syntactic properties of symbols and their semantic relationships to the objects they denote, were essential for all later studies of definability. Moreover, in that publication Peano was introducing logical symbolism that quickly became famous for its precision and infamous for its opacity. Over decades it was transformed into the familiar notation used later in the present paper.

Peano became a center of mathematical controversy. He pointed out and corrected errors and oversights in various texts, and introduced a number of our now standard delicate arguments in analysis. In 1891 he published a polemical interchange with the noted Turin algebraic geometer Corrado Segre about rigor in geometry. Segre favored leniency, to foster intuitive discovery and formulation of general results, leaving exceptional cases to later study. Peano roared that a seemingly general statement that is sometimes false cannot be a mathematical theorem. Bertrand Russell reported that Peano always got the best of the arguments following papers presented at the 1900 Paris congress of philosophers. In the introduction to his 1894 paper [**35**], to be discussed next, Peano railed about geometry texts that began with vague descriptions of numerous concepts such as space, homogeneity, and unboundedness. There and in closing, Peano indicated the relevance of his work for improving elementary instruction.[7]

In [**35**] Peano introduced the use of *direct motion* to replace congruence as a primitive concept in Euclidean geometry. A geometric transformation, this sort of motion

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[5][**30**, p. 17]; see also [**14**, §6].

[6]The identity of the supervisor of Peano's doctoral study seems uncertain. In [**33**] Peano defined coplanarity in terms of collinearity, and thus avoided listing it as a primitive notion. He introduced notation and terminology that enabled him to analyze such ideas more deeply than Pasch.

[7]See [**16**, Chaps. 3–5], [**22**], [**13**], and [**43**, pp. 232–233]. Russell wrote that his encounter with Peano changed his intellectual life; later he achieved worldwide renown in logic and philosophy. Pasch also complained in his 1894 celebratory address [**31**, pp. 12, 17] that most textbooks were still inadequate. He emphasized their vague treatment of concepts such as space and dimension. In 1899, Peano's associate Giuseppe Ingrami published an elementary text [**12**] based on Peano's approach to geometry.

does not involve time: only the initial and final positions of a figure are relevant. Motions are isometries: they preserve distance. Those that also preserve orientation are called direct. Figures can be defined as *congruent* if some direct motion maps one to the other. (In contrast, indirect motions reverse orientation; figures related by indirect motions can be called anticongruent.) Some of Peano's axioms stated geometric properties of motions: for example, should A and B be distinct points, P a point between them, and m a motion, then the image mP must lie between mA and mB. Others, however, would now be called group-theoretic: the identity should be a motion, motions should be bijective, and their inverses should be motions.

5. PIERI AND MOTIONS. Mario Pieri was the third of eight children of a lawyer in Lucca, Italy. He was schooled there and in Bologna. Pieri attended university in Bologna and Pisa, where he earned the doctorate in 1884 in algebraic and differential geometry, under the supervision of Luigi Bianchi. Two years later, Pieri became professor of projective and descriptive geometry at the Royal Military Academy in Turin. In 1888 he was also appointed assistant at the nearby university. Soon after, Corrado Segre suggested that Pieri translate G. K. C. von Staudt's fundamental 1847 work [**47**] on the projective geometry that underlies those subjects. Pieri published that in 1889. Evidently he, like Pasch, became intrigued with its logic: he returned to study it again and again. Pieri also continued to work in algebraic geometry, and became particularly noted in that field for his methods of enumerating solutions of systems of algebraic equations.

Pieri's senior colleague Giuseppe Peano was already investigating deep questions in foundations, and in an 1890 paper [**34**] repaired a lapse in Staudt's work. During the next two decades, in several major studies in foundations of geometry, Pieri used, refined, and publicized Peano's logical techniques. A series of Pieri's papers culminated in the 1898 memoir [**37**], the first full axiomatization of projective geometry. This employed as primitive only the concepts of point and of the line joining two points, and it covered all dimensions.[8] Pieri presented that work and later axiomatic studies as *hypothetical-deductive systems*. He and Peano's collaborator Alessandro Padoa formally introduced in 1900 this explicit formulation of Peano's idea that the primitive concepts may be interpreted arbitrarily, as long as the interpretations satisfy the axioms. That framework was adopted by postulate theorists during 1900–1925, particularly Edward V. Huntington, who precisely echoed the Italians' formulation. Recent historical analysis has confirmed that this approach turned into today's version of the axiomatic method.[9]

Following Peano's lead, Pieri pursued deeply the use of direct motion as a primitive concept. His 1900 *Point and Motion* memoir [**39**] axiomatized a large part of three-dimensional geometry common to the Euclidean and hyperbolic. He employed only two primitive concepts, *point* and *direct motion*. The following definitions were central:

[8] Segre had suggested in 1892 that an axiomatic study was needed to establish a synthetic approach to higher-dimensional projective geometry. Pieri's colleague Gino Fano indicated then [**4**, p. 106] that it was necessary to determine what properties of flat subspaces were critical for that development. Pieri started an 1896 paper [**36**] with the comment that higher-dimensional projective geometry was still a controversial subject.

[9] See [**39**, Preface], [**40**, §III], [**24**], [**11**, pp. 288–290], [**14**, §6], and [**44**, §2]. Born in Venice, Padoa (1868–1937) earned the doctorate from the University of Turin in 1895 under Peano's supervision, with a dissertation on foundations of geometry. An especially effective expositor, Padoa remained one of Peano's closest collaborators. His career included various positions in middle schools, universities, and the naval institute, mostly in Genoa.

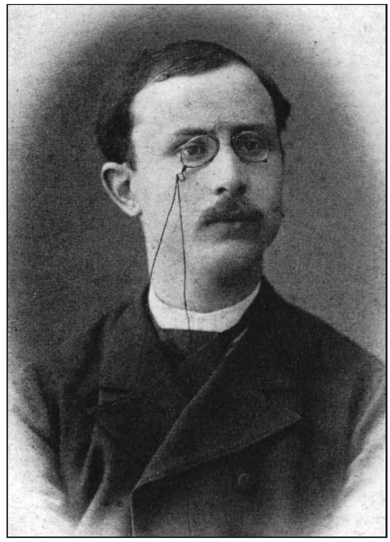

Figure 2. Mario Pieri (1860–1913).

- Three points are called *collinear* if they are fixed by some nontrivial direct motion. (Think of an axial rotation.)
- Points M and P are called *equidistant* from a point O if some direct motion maps M to P but leaves O fixed.
- A point is said to *lie midway between* two others if it is collinear with and equidistant from them.
- A point Q is said to *lie somewhere between* two points P and R if it is collinear with them and lies midway between two points M and N such that the pairs M, P and N, P are equidistant from a point O midway between P and R. Figure 3 displays this ingenious definition.

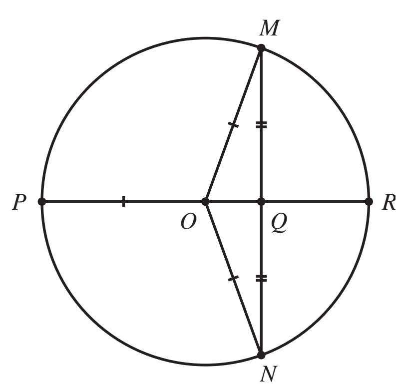

Figure 3. Pieri's 1900 definition of *Q lies somewhere between P and R*.

Pieri adapted his definition of collinearity from the 1679 work of G. W. Leibniz [**20**, p. 147], which was then under serious study by scholars in Peano's group. Pieri formulated complicated axioms in terms of these primitive concepts, and provided full proofs of all his theorems. He did not include axioms about completeness of lines, and thus stopped short of proving that the points on a given line enjoy all the properties of the real numbers.[10]

6. INTERLUDE.

Mario Pieri's *Point and Motion* memoir [**39**] was published almost simultaneously with David Hilbert's 1899 masterpiece, *Foundations of Geometry* [**10**], the most famous treatment of this subject. The two were independent and very different. With a single introductory sentence, "... [this work] is tantamount to the logical analysis of our intuition of space," Hilbert claimed that his underlying philosophy agreed with that of Moritz Pasch, as described here in Section 3. But the abstract approach he actually followed was like the Italians'. Reviewing Hilbert's book, Hans Freudenthal wrote, "What was expressly formulated by [Pieri and] Padoa was tied up in Hilbert's work with the mathematically important facts. ..." Freudenthal lauded "the convincing power of a philosophy that is not preached as a program, but is only the silent background. ... This thoroughly and profoundly elaborated piece of axiomatic workmanship was infinitely more persuasive than programmatic and philosophical speculations on space and axioms could ever be." Hilbert formulated some axioms in terms of defined concepts, which permitted simple exposition. He presented proofs in familiar style, with much left unsaid—readers could supply that if they wished. Pieri showed how to formulate axioms, no matter how complicated, solely in terms of the primitive concepts, and how to provide all details of proofs. Hilbert's presentation popularized the subject. Pieri's never attracted great acclaim but, as you will see, it provided a path for significant later research.[11]

In 1904 Oswald Veblen proposed an axiomatization [**54**] of Euclidean geometry that regarded only *point* and *betweenness* as primitive concepts. His axioms were simpler than Pieri's or Hilbert's. Veblen followed Pasch [**30**] in using a projective polar system to define Euclidean congruence and then equidistance and motion. In 1907, however, Federigo Enriques noted [**2**, §6] that Veblen's polar system was not uniquely determined: it seemed also to be undefined, and thus a very complicated primitive concept. In 1911, Veblen published another axiomatization [**55**] that avoided this problem.[12]

7. PIERI'S *POINT AND SPHERE* MEMOIR.

After a long struggle Mario Pieri obtained appointment as university professor in 1900 at Catania on the Italian island of Sicily. There he completed his 1908 *Point and Sphere* memoir [**41**], a full axiomatization of Euclidean geometry based solely on the primitive concepts *point* and *equidistance* of two points N and P from a third point O, written $ON = OP$. He

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[10] Introducing [**39**], and in his 1900 Paris address [**40**, §III], Pieri quoted Pasch's 1894 comment. Pieri contrasted Pasch's and Peano's incisive work with the proliferation of unexplained concepts in Wilhelm Killing's 1893–1898 text [**17**], including space, occupying a space, and so on. Pieri displayed considerable interest in reforming elementary instruction, particularly in his review [**38**] of Giuseppe Ingrami's text and in the introduction and appendix to his 1908 *Point and Sphere* memoir [**41**].

[11] [**10**, Intro.], [**7**, pp. 618–621]; see also [**53**, Sec. 6.1]. Hilbert (1862–1943) was born near Königsberg, Prussia (now Kaliningrad, Russia); his father was a judge. Hilbert earned the Ph.D. at the university there in 1884 under the supervision of Ferdinand von Lindemann. By 1899 Hilbert was a senior professor at Göttingen and had achieved worldwide acclaim. Freudenthal reported that Hilbert's book did attract some controversy, but not for long.

[12] Veblen (1880–1960) was born in Decorah, Iowa; his father was a professor. Veblen earned the Ph.D. at the University of Chicago in 1903 under the supervision of Eliakim H. Moore, the leader of the American postulate theorists. Veblen's 1904 paper stemmed from his dissertation. Later, Veblen would play a leading role in the development of American mathematics.

had suggested this possibility already in 1900 [**39**, Preface].[13] Pieri used the following definitions (letters N to R may refer to points in Figure 4):

- N is said to lie *on the sphere* P_O *through* P *about* O if $ON = OP$.
- If $O \neq Q$, then P is called *collinear with* O and Q if P_O intersects P_Q only at P. (Pieri adapted this definition from G. W. Leibniz [**20**, Part IV, pp. 185, 189].)
- Q is called a *reflection of* O *over* P, and P is said to *lie midway between* O and Q, if $PO = PQ$ and Q is collinear with O and P.
- Two spheres are called *congruent* if the points on one are related to those on the other by reflection over some single point.
- Point pairs O, P and Q, R are called *congruent* if R lies on a sphere about Q congruent to P_O.
- An *isometry* is a point transformation that preserves congruence of pairs.
- A *direct motion* is the composition of an isometry with itself.[14]

Pieri then proceeded as in [**39**]. In particular, he adopted verbatim his earlier definition of betweenness, thus achieving one solely in terms of equidistance of two points from a third. His axioms were frightfully complicated. But except for his Archimedean and continuity axioms, they would now be called *first-order:* they can be phrased solely in terms of the primitive concepts without using any set theory. Further, Pieri again published all details of his proofs: the translation in [**23**, Chap. 3] is 111 pages long! But his memoir, overshadowed by David Hilbert's book [**10**], received little acclaim, not even one review.

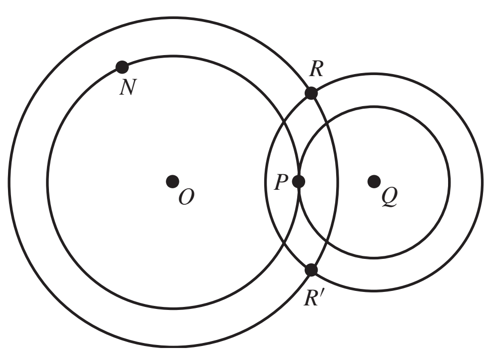

Figure 4. Pieri's 1908 definition of *collinearity*: P is collinear with O and Q; R is not.

Pieri's untimely death in 1913 and the following decades of war and economic and political turmoil probably contributed to this neglect. *Point and Sphere* shined through that fog once in 1915, as a Polish translation, and a decade later in the work of the

[13]Using ternary equidistance $ON = OP$ as primitive rather than quaternary segment congruence $MN = OP$ occurred to Alessandro Padoa independently at about the same time. He understood Pieri's intention only in Paris, just before his own talk [**25**] to the International Congress of Mathematicians. That preliminary report explicitly mentioned only segment congruence. But close reading reveals that Padoa employed ternary equidistance wherever he could. In a 1901 letter [**26**], Padoa explained this situation to Pieri with great deference. In effect, Padoa ceded this research area to Pieri.

[14]In the intended interpretation, an isometry is a direct motion just when it is a "square." To see this, recall that some isometries may be indirect, reversing orientation; their "squares" must be direct. Moreover, every direct motion is a screw—a composition of a rotation about an axis and a translation along it—and hence is the "square" of the screw with half the angle and half the vector.

Polish logician Alfred Tarski. The present author does not know exactly how the thread of this story wound in that direction. Ongoing historical research suggests that this was due to the influence of Giuseppe Peano among some Polish mathematicians.

8. TARSKI'S SYSTEM OF GEOMETRY. Alfred Tarski was born in Warsaw, Poland, then part of Russia. The family was Jewish and secular, involved in business. There were two children: Alfred had a younger brother. Schooled during World War I, Alfred became a Polish nationalist. He entered the university in Warsaw afterward, and in 1924 earned the doctorate in logic under the supervision of Stanisław Leśniewski. The economy and antisemitism made it hard to find a job. In 1926, starting on the path to become one of the world's top logicians, Alfred Tarski was a university assistant and high-school teacher in Warsaw.[15]

At this time, Tarski's research emphasis included application of logical techniques to geometric problems. Studying axiomatic presentations of geometry, he adopted the refinements of the axiomatic method introduced by Giuseppe Peano, Alessandro Padoa, and Mario Pieri.[16] Tarski was also beginning to emphasize *first-order* logic. This framework employs just the logical symbols for equality, Boolean connectives, variables ranging over individuals in a specific domain, the existential and universal quantifiers $\exists$ and $\forall$, and selected constants, relations, and operations in and on that domain. Thus it minimizes the use of set-theoretic notions in the logical framework. Pieri's 1908 *Point and Sphere* memoir fit into that framework. Reporting a conversation with Tarski, Steven Givant wrote,

> Tarski was critical of Hilbert's axiom system [**10**]... [and] preferred Pieri's system [**41**], where the logical structure and the complexity of the axioms were more transparent.

Tarski developed and presented his own axiomatization in a 1926–1927 course at Warsaw University. According to his later collaborator Lesław Szczerba, "the system that Tarski presented in this course was designed after" Pieri's 1908 memoir.[17]

Tarski's primitive concepts were *point* and two relations among points, the ternary relation expressing *betweenness* and the quaternary one expressing *congruence* of point pairs. Oswald Veblen had completed an axiomatization [55] based on those three concepts in 1911. With these slightly more complex primitives, Tarski was able to greatly simplify Pieri's 1908 axioms. Tarski's axioms were two-dimensional, but he noted that they could be easily modified for use in three or higher dimensions without loss of simplicity.[18] As continuity axioms, Tarski used all first-order instances of Pieri's second-order axiom. All Tarski's axioms except those for continuity had $\forall\exists$ form, with all quantifiers at the beginning, universal preceding existential. Their total length was less than that of Pieri's single most complicated axiom. That one, Pieri's version of the Pasch axiom, had form $\forall\exists\forall\exists$. (A standard procedure will convert any first-order sentence to a logically equivalent form with all quantifiers at the

[15] For biographical information, consult [**5**]. Before 1936, Tarski published several works about high-school geometry or addressed to high-school teachers [**23**, Sec. 5.2]. Their relationship to Tarski's mathematical research is under investigation by Andrew McFarland and the present author.

[16] See [**14**] and [**44**].

[17] See [**8**, p. 50] and [**48**, p. 908]. Whether Tarski consulted Pieri's original memoir [**41**] or its 1915 Polish translation is an open historical question. In the latter case, Tarski's work would be linked directly to the Polish Mianowski foundation, which supported the translation, and probably to Peano's contacts with Polish intellectuals during 1900–1915.

[18] See [**50**, Sec. 3.6], and [**51**, footnote 5].

Figure 5. Alfred Tarski (1901–1983).

beginning. The number of alternations of quantifier types then provides a measure of complexity.[19])

Tarski proved that the structures satisfying his axioms are those isomorphic with coordinate planes over real-closed ordered fields—ordered fields in which every polynomial with odd degree has a root, and whose nonnegative elements are all squares. Because of economic and political turmoil and war, Tarski's system was not broadly publicized until his 1959 summary [**51**], *What Is Elementary Geometry?* In lectures, Tarski followed Pieri's practice of full disclosure of all proofs, but they remained unpublished commercially until the appearance of [**45**] in 1983.[20] The earlier formulation of the system, though, enabled much deeper research into provability, decidability, and definability in geometry.

9. NONDEFINABILITY. Consider elementary geometry based on *congruence* δ of point pairs and *betweenness* β of triples, like Alfred Tarski's system. Formulas such as βABC and $\delta ABCD$ should be read, "B lies between A and C" and "pairs A, B and C, D are congruent." As described here, Mario Pieri had shown in [**41**] how to construct a formula φABC involving just δ, and to prove $(\forall A, B, C)[\beta ABC \Leftrightarrow \varphi ABC]$. That is, φ *characterizes* β, and β is definable from δ; it could thus be eliminated from the list of primitive concepts. In [**54**] Oswald Veblen had claimed the reverse: that he had defined δ from β. But in [**2**] Federigo Enriques had objected: Veblen had not. Could δ *be defined from* β *at all?*

Settling this question required a precise definition of *definition*. That was achieved by first adopting as standard some axiom system such as Tarski's, assumed consistent. If ν is a concept and Φ a family of concepts defined in that system, then a first-order phrase mentioning only the concepts in Φ should be called a *definition* of ν in terms

[19] See [**19**]. In 2008 Victor Pambuccian [**28**] showed how to construct an axiom system equivalent to Pieri's, with the same two primitives, in which all axioms except those for continuity have $\forall\exists$ form. He also referred there to others' recent work on equivalent systems with those primitives.

[20] Many of Tarski's students and colleagues contributed to proofs published there.

of Φ if it provably characterizes ν in the standard system. In the 1935 study [**49**, §1], after considering definitions in general, Tarski noted that indeed, betweenness β *cannot* serve as the sole primitive relation in an axiomatization of elementary geometry with variables ranging over points. His discussion suggests the following argument, based on a technique introduced in 1900 by Alessandro Padoa [**24**, Sec. 16]: any affine transformation that is not a similarity would preserve β, and thus also any concept defined by a first-order phrase solely in terms of β, but it would not preserve the congruence relation δ.

To make that argument precise, consider such a transformation $A \rightsquigarrow A'$ that is represented by a formula $\alpha AA'$ of the standard system. A convenient choice is the shear that would be described by the equations $x' = x + y$ and $y' = y$ using a Cartesian coordinate system based on an origin O and some unit points X and Y. The formula α should include the clause $\delta OXOY$ and the requirement that $\angle XOY$ be a right angle. (The simplest α might describe a construction of A' from A using ΔXOY and various other triangles.) These formulas can then be proved (see Figure 6):

$$\alpha OO' \;\&\; \alpha XX' \;\&\; \alpha YY' \Rightarrow O = O' \;\&\; X = X' \;\&\; \delta OXOY \;\&\; \neg\, \delta OXOY' \tag{1}$$

$$\alpha AA' \;\&\; \alpha BB' \;\&\; \alpha CC' \Rightarrow (\beta ABC \Leftrightarrow \beta A'B'C'). \tag{2}$$

Now suppose δ *were* definable from β: for some formula φ involving just β the sentence $\forall ABCD(\varphi ABCD \Leftrightarrow \delta ABCD)$ should be provable. Rules of logic would yield a proof of the analog of (2) with φ in place of β, and thus of

$$\alpha AA' \;\&\; \alpha BB' \;\&\; \alpha CC' \;\&\; \alpha DD' \Rightarrow (\delta ABCD \Leftrightarrow \delta A'B'C'D'). \tag{3}$$

Substituting O, X, O, Y for A, B, C, D would then yield a formula that contradicts (1).

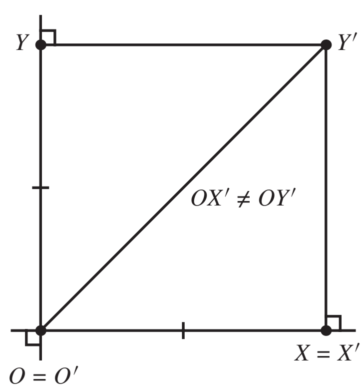

Figure 6. Shear α: $O, X, Y \rightsquigarrow O', X', Y'$.

That same year, Adolf Lindenbaum and Tarski presented a related argument that has the following consequence: no family Φ, however large, of binary relations definable in the standard system can serve as the family of all primitive concepts. Thus Pieri's selection of ternary equidistance as the sole primitive concept was in a sense *optimal*. The argument proceeded by reasoning about the real Euclidean plane $\mathbb{R}^2$, with the usual betweenness and congruence relations. It is a model of the standard system of geometry. The same symbols are used for formulas in the system and for their interpretations in the model. The argument is based on the automorphism group, emphasized by Felix Klein, as mentioned in Section 2; the group consists of the similarity transformations.[21]

First, consider a binary relation on $\mathbb{R}^2$ represented by a formula ρAB. It must be invariant under any similarity $A \rightsquigarrow A^\sigma$ of $\mathbb{R}^2$, because the betweenness and congruence relations are invariant. That is, for any $A, B \in \mathbb{R}^2$, statements ρAB and $\rho A^\sigma B^\sigma$ are equivalent: either both true or both false. Now, given two pairs A, B and A', B' of points in $\mathbb{R}^2$, either both coincident or both distinct, there is a similarity σ such that $A^\sigma = A'$ and $B^\sigma = B'$. Consequently, statements ρAA and $\rho A'A'$ are equivalent, as are ρAB and $\rho A'B'$. Thus, either (1) ρAA is true for all $A \in \mathbb{R}^2$ or (2) it is false for all such A; and moreover, either (a) ρAB is true for all pairs A, B of distinct points in $\mathbb{R}^2$, or (b) it is false for all such pairs. In the four cases (1a), (1b), (2a), and (2b), ρ represents the universal, equality, inequality, and empty relation, respectively. Therefore, a set Φ of representable binary relations on $\mathbb{R}^2$ can contain only those four. But they are invariant under *every* transformation. Since neither the betweenness nor the congruence relation has that property, these cannot be defined solely in terms of the formulas representing relations in Φ.

Tarski's work has led to related studies, many reported in [**45**]: for example, what other ternary relations suffice as the sole primitive relation? More recently, Victor Pambuccian has investigated the effect of strengthening the underlying logic to permit conjunctions $\&_{m,n}\varphi_{m,n}$ of infinite families of open sentences $\varphi_{m,n}$ that depend on natural numbers m, n. In 1990 he discovered a startling fact [**27**]: with that logic, a single binary relation υ can be used as the sole primitive relation for a system of geometry very closely related to Tarski's! This relation υPQ holds for points P and Q just when the distance PQ is equal to 1. Incorporating υ into Tarski's system adds theorems about specific distances, and restricts the automorphisms to the group of isometries.[22]

Pambuccian considered for each m, n the auxiliary relation $\upsilon_{m,n}PQ$ that says $PQ = m/2^n$. This relation can be defined solely in terms of υPQ by a complicated first-order formula that describes some familiar geometric constructions. He then proved that P, Q is congruent to another point pair R, S just when

$$\&_{m,n}\Big[\big[\exists\, T[\upsilon_{m,n}PT \;\&\; \upsilon_{m,n}QT] \Rightarrow \exists\, U[\upsilon_{m,n}RU \;\&\; \upsilon_{m,n}SU]\big]$$
$$\&\big[\exists T[\upsilon_{m,n}RT \;\&\; \upsilon_{m,n}ST] \Rightarrow \exists\, U[\upsilon_{m,n}PU \;\&\; \upsilon_{m,n}QU]\big]\Big].$$

The first implication fails for some m, n just when $PQ < RS$: see Figure 7. The formula just displayed is a definition, in the strengthened logical system, of congruence of point pairs. That yields a definition of Pieri's single primitive relation, ternary equidistance, solely in terms of υ. Therefore, with the new logic, υ can also serve as the single primitive relation. But the geometry is not new: the constructions that Pambuccian employed in 1990 to define $\upsilon_{m,n}$ in terms of υ had already been used by Pieri in 1908 to analyze the continuity of a line![23]

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[21]See [**21**, §1]. The argument in the following paragraph is adapted from [**45**, pp. 285–287]. This result would still hold should Φ also contain singulary relations σ, or properties: just replace each such σ by a new binary relation ρ such that ρPQ stands for σP.

[22]Thus, the preceding paragraph's argument, that the primitive relations cannot all be binary, fails for the augmented system. But Raphael M. Robinson proved in [**42**], by a different method, that the result still holds, provided the underlying logic remains standard.

[23]See [**41**, §IV, proposition P18, and §VIII, proposition P21ff.]. For information on logic with infinite conjunctions, consult [**15**]. The displayed sentence says that the following statement holds for all natural numbers m and n: whenever there is a point T at distance $m/2^n$ from each of P, Q then there is a point U at that distance from each of R and S, and conversely. Victor Pambuccian (1959–) earned the doctorate from the University of Michigan in 1993 under the supervision of Andreas Blass, with a dissertation on foundations of geometry. He is the author of many axiomatic studies related to questions in the present paper.

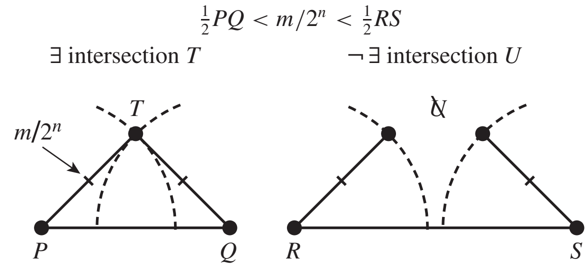

Figure 7. Pambuccian's 1990–1991 definition of $PQ < RS$.

10. CONCLUSION.

The pioneering work of Giuseppe Peano during 1889–1894 on the logic underlying geometry and on the use of direct motion as a fundamental geometric idea led from the systems of Euclid and Moritz Pasch straight to Mario Pieri's detailed 1900 and 1908 axiomatizations of geometry. Pieri formalized his presentations as hypothetical-deductive systems, which became our familiar setting for axiomatic studies. His choice of primitive concepts was spare, he relied on set theory only for continuity considerations, and he published all details of his proofs. During the 1920s Alfred Tarski followed Pieri's approach to achieve a surprisingly efficient first-order axiomatization of Euclidean geometry, which has become a standard of comparison for work in foundations of geometry. Tarski formulated in the 1930s a theory of first-order definitions, with which he showed that Pieri's choice of primitives was in a sense optimal. In the 1990s, Victor Pambuccian, using geometry that would have been familiar to Pieri, showed that some greater economy could be achieved, but only by strengthening the underlying logic and slightly changing the geometry under consideration. Research in this field continues today.

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