Canadian Society for History and Philosophy of Mathematics
Société canadienne d'histoire et de philosophie des mathématiques

Annual Meeting, 30 May 2010

Definitions and Nondefinability in Geometry: Legacies of Mario Pieri and Alfred Tarski

James T. Smith, Professor Emeritus
San Francisco State University

I thank the Society sincerely for the honor of being invited here, and acknowledge inspiration by Elena Anne Marchisotto and suggestions from Victor Pambuccian.

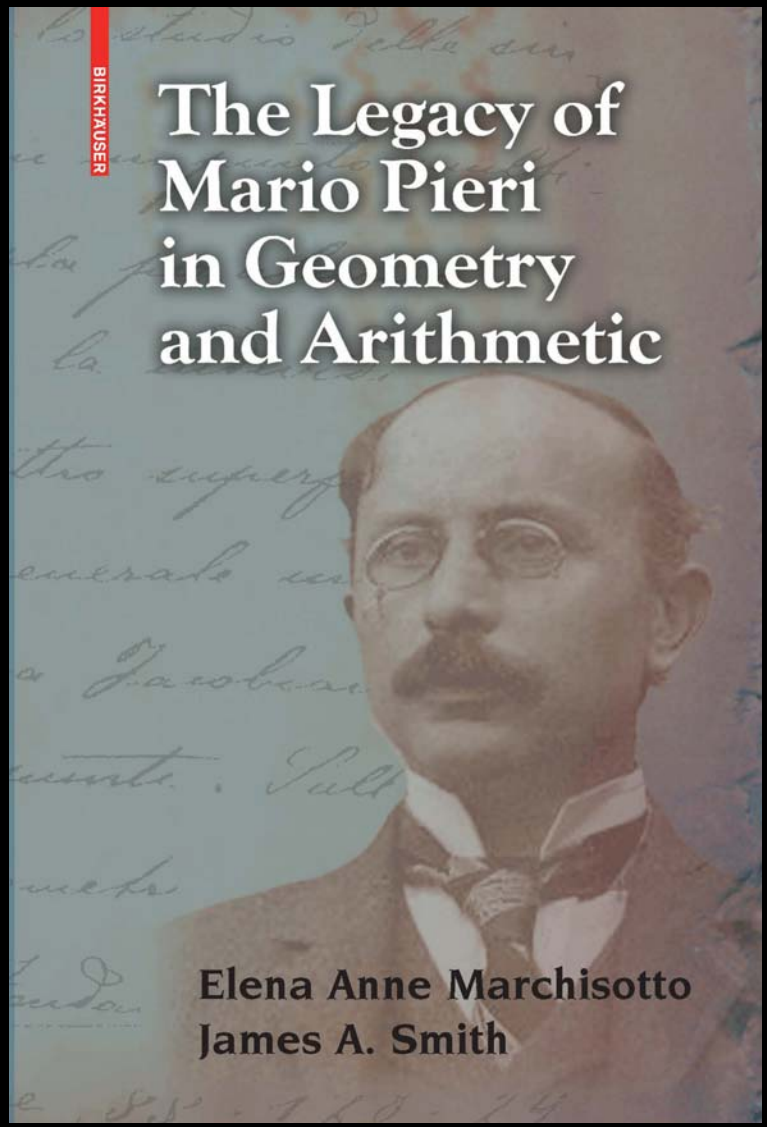
Definitions and Nondefinability in Geometry

James T. Smith

- **This presentation is about**
  - **Euclidean geometry as an axiomatic system,**
  - **definitions in geometry, and**
  - **definability in geometry, and emphasizes**
  - **contributions of Mario Pieri and Alfred Tarski.**

It's adapted from and expands on Marchisotto & Smith 2007.

A full version will appear soon in the *American Mathematical Monthly*.



Marchisotto & Smith 2007

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James T. Smith

- **Pasch**
- **Peano**
- **Pieri**
- **Veblen & Enriques**
- **Pieri's *Point & Sphere***
- **Tarski**
- **Nondefinability**

Pasch

♦ During the 1800s ...

Moritz PASCH (1843–1930) portrait

- **saw the need to repair Euclid,**
- **to firmly ground the popular projective methods.**

Pasch's 1882 ***Vorlesungen über neuere Geometrie*** **title page**

During the 1800s

The relationship of various theories became very involved: for example,

- **Euclidean and non-Euclidean geometry can each be extended to projective geometry;**
- **we can introduce projective coordinates;**
- **use a *polar system* to measure distance; and**
- **reconstruct Euclidean or non-Euclidean geometry from different polar systems.**

Algebraic geometry relied more and more on projective methods.

This complexity led to *new emphasis on axiomatics*.

Pasch

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  - **saw the need to repair Euclid,**
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Pasch's 1882 ***Vorlesungen über neuere Geometrie*** title page

Alexandra Zirock

VORLESUNGEN

ÜBER

NEUERE GEOMETRIE

VON

DR. MORITZ PASCH
PROFESSOR AN DER UNIVERSITÄT ZU GIESSEN

ZWEITE MIT ZUSÄTZEN VERSEHENE AUSGABE

MIT 72 FIGUREN IM TEXT

LEIPZIG UND BERLIN
DRUCK UND VERLAG VON B. G. TEUBNER
1912

Pasch
1882

Pasch 1882

♦ • **developed incidence and congruence geometry,**

• **extended it to projective space, then showed how**

• **to select a Euclidean or non-Euclidean polar system.**

Its primitive concepts were

- ***point***
- ***segment between*** **two points**
- ***coplanarity*** **of a point set**
- ***congruence*** **of point sets**

Example definitions:

- **Three points are *collinear* if they are not distinct or one lies between the other two.**
- **The *line* determined by two distinct points is the set of points collinear with them.**

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- ♦ Peano
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- Nondefinability

Peano

- **Giuseppe PEANO (1858–1932)** portrait
  - **as a junior professor at the University of Torino undertook to reformulate with utmost precision *all* of pure mathematics!**

Peano's 1889 book title page

- **freed the axiomatic method from references to the real world.** page 24

Peano's 1894 paper ***Sui fondamenti della geometria***

Giuseppe Peano
(1858–1932)

Peano

Giuseppe PEANO (1858–1932) **portrait**

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Peano's 1894 paper ***Sui fondamenti della geometria***

Alexander Ziwet

I

PRINCIPII DI GEOMETRIA

LOGICAMENTE ESPOSTI

SAGGIO

DI

GIUSEPPE PEANO

Professore nella R. Accademia militare
libero docente nella R. Università di Torino.

TORINO
FRATELLI BOCCA EDITORI
LIBRAI DI S. M. IL RE D'ITALIA

SUCCURSALI
ROMA
Via del Corso, 216-217
FIRENZE
Via Cerretani, 8

DEPOSITI
PALERMO
Università, 12
(N. Carosio)
MESSINA
(Daly)
CATANIA
S. Maria al Ros.°, 23
(N. Carosio)

1889

Peano
1889

Peano

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NOTE

§ 1.

P2. Si legge « la proposizione $a = b$ è equivalente alla $b = a$ ». Volendo, invece dei punti, usare le parentesi, si dovrebbe scrivere
$(a = b) = (b = a)$.

P3. « Dall'insieme delle proposizioni $a = b$, e $b = c$ si deduce la $a = c$ ». I punti stanno per indicare l'aggruppamento
$((a = b)(b = c)) \supset (a = c)$.

Le proposizioni 1, 2, 3 esprimono le proprietà caratteristiche di ogni identità. Esse sono fra loro irreduttibili.

P4. « Se a e b sono punti, allora ab indica una classe di punti (figura) ».

P5. « Se a, b, c, d sono punti, a coincide con b, c con d, allora il segmento $\overset{\frown}{ab}$ è identico a $\overset{\frown}{cd}$ ».

Le proposizioni 4 e 5 dicono che « il segmento ab è una classe di punti, determinata dati i due punti a e b ».

Si ha così una categoria di enti, chiamati punti. Questi enti non sono definiti. Inoltre, dati tre punti, si considera una relazione fra essi, indicata colla scrittura $c \in ab$, la quale relazione non è parimenti definita. Il lettore può intendere col segno 1 una categoria qualunque di enti, e con $c \in ab$ una relazione qualunque fra tre enti di quella categoria; avranno sempre valore tutte le definizioni che seguono (§ 2), e sussisteranno tutte le proposizioni del § 3. Dipendentemente dal significato attribuito ai segni non definiti 1 e $c \in ab$, potranno essere soddisfatti, oppure no, gli assiomi. Se un certo gruppo di assiomi è verificato, saranno pure vere tutte le proposizioni che si deducono, non essendo queste proposizioni che trasformazioni di quegli assiomi e delle definizioni.

Peano 1889, 24

There is a category of things called points. These things are not defined. Moreover, given three points, a relation between them is considered, indicated by $c \in ab$, which is likewise undefined. The reader can denote by the symbol 1 an arbitrary category of things and by $c \in ab$ an arbitrary relation between three things in this category; all the definitions that follow (§2) will always be valid, and all the propositions of §3. Depending on the significance attributed to the undefined symbols 1 and $c \in ab$, the axioms can be satisfied or not. If a certain group of axioms is verified, then all the propositions that are deduced from them will be equally true, being merely transformations of these axioms and definitions.

Peano

Giuseppe PEANO (1858–1932) portrait

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Peano's 1889 book title page

- **freed the axiomatic method from references to the real world.** page 24

♦ **Peano's 1894 paper *Sui fondamenti della geometria***

Peano 1894:
Sui fondamenti della geometria

- **introduced the use of (direct) *motion* as a primitive concept.**
- **A geometric transformation, it *does not involve time*.**
- **It replaced *congruence*.**

Figures are then *defined* as congruent if a motion maps one to the other.

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- **Pieri's *Point & Sphere***
- **Tarski**
- **Nondefinability**

Pieri

- **Early career**

Pieri's 1900 *Monograph on Point and Motion*

Pieri's Early Career

- **Mario PIERI (1860–1913)**

  - **Doctorate at Pisa in algebraic geometry, then**
  - **Professor at the Turin military academy and assistant at the University.**

  portrait

  - **Influenced by C. Segre and Peano to study foundations of projective geometry,**
  - **Pieri achieved its first complete axiomatization, then**
  - **turned to absolute (Euclidean ∩ hyperbolic) geometry,**
  - **emphasizing (direct) motion as a primitive concept.**

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MEMORIE

DELLA

REALE ACCADEMIA

DELLE SCIENZE

DI TORINO

SERIE SECONDA

TOMO XLIX

TORINO

CARLO CLAUSEN

Librajo della R. Accademia delle Scienze

1900

DELLA

GEOMETRIA ELEMENTARE

COME SISTEMA IPOTETICO DEDUTTIVO

MONOGRAFIA DEL PUNTO E DEL MOTO

DI

MARIO PIERI

LIBERO DOCENTE ALL'UNIVERSITÀ DI TORINO

Approvata nell'Adunanza del 14 Maggio 1899.

PREFAZIONE

" Die ersten Begriffe, mit denen eine Wissenschaft beginnt, müssen klar und auf die kleinste Zahl zurückgeführt sein. Nur dann können sie für das Lehrgebäude eine feste und genügende Grundlage bilden „. N. I. LOBATSCHEFSKIJ, *Ueber die Anfangsgründe der Geometrie* (Uebrs. ENGEL).

" Les mathématiques pures progressent à mesure que les problèmes connus sont approfondis en détail d'après des méthodes nouvelles. A mesure que nous comprenons mieux les anciens problèmes, les nouveaux se présentent d'eux mêmes „. F. KLEIN, *Discours prononcé à Vienne* le 27 septembre 1894 (Trad. LAUGEL).

Per sistema ipotetico-deduttivo intendiamo qualunque dottrina puramente deduttiva — o scienza di ragionamento — la quale non solo distingua organicamente i giudizi a priori, o *primitivi*, da quelli derivati, o *dedotti*, e insomma gli assiomi e postulati dai teoremi; ma così ancora e nella stessa misura disponga le varie nozioni intorno a cui versano questi giudizi, segnalando perciò le idee madri, *primitive*, o indecomposte, e mantenendole ben distinte da quelle che ne sono riproduzioni e derivazioni formali o possono aversi per tali, e che insomma risultano effettivamente composte mediante le prime combinate fra loro e con le categorie della Logica. Le due distinzioni sono in verità molto affini; e la seconda non è meno antica dell'altra, né par che le spetti un valore molto diverso: ma con tutto ciò non l'è stata riconosciuta praticamente un'eguale importanza dai matematici prima dei nostri tempi (*). Invero si cercò per lo più di ridurre in minor

(*) Se per definizione s'intenda una pura e semplice imposizione di nomi a cose già note od acquisite al sistema, le idee primitive saranno i concetti *non definiti*. Ma il " definire „ è

Pieri 1900

Pieri 1900: *Point and Motion*

- **A translation will appear in Marchisotto et al. 2011(?).**

- **It developed geometry from just *two* primitive concepts, *point* and *motion*, and provided *complete proofs* of all theorems.**

- **Pieri framed it as a *hypothetical-deductive system* (= our *axiomatic theory*), following Peano's approach, as publicized by Pieri and Alessandro PADOA.**

- **Some critical definitions**

Pieri 1900 Definitions

P*, *Q*, *R* are *collinear ⇔ (∃ nontrivial motion μ)
[$\mu P = P$ & $\mu Q = Q$ & $\mu R = R$],
following Leibniz. Motions must be direct!

P*, *Q* are *equidistant* from *R ⇔ (∃ μ)[$\mu P = Q$ & $\mu R = R$]

Q* is *midway between* *P*, *R ⇔ ***Q* is collinear with & equidistant from *P*, *R***

Q* is *somewhere between* *P*, *R ⇔ see figure 1

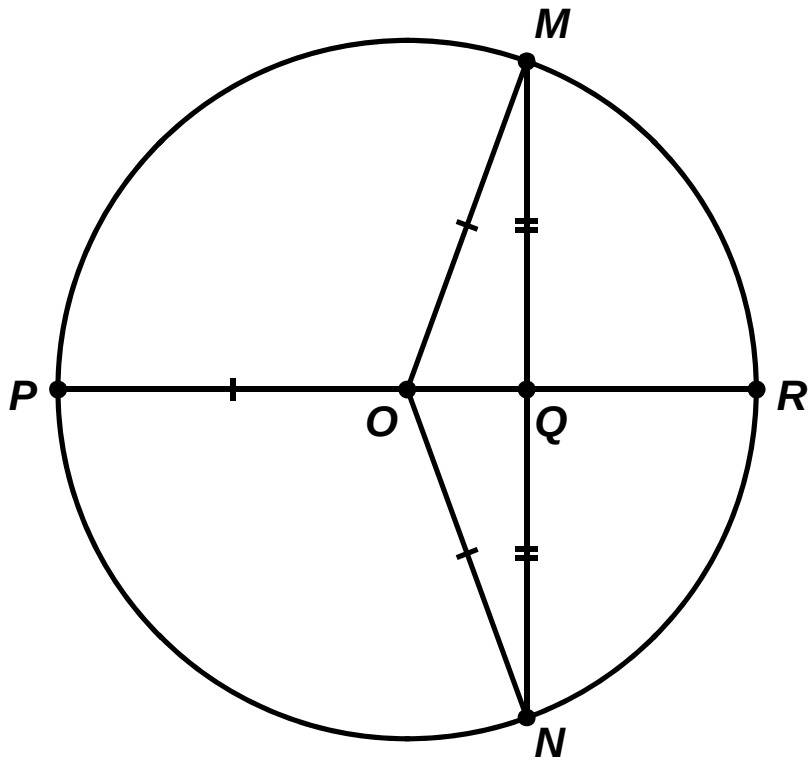

Figure 1

Pieri's 1900 definition of

Q lying somewhere between P, R

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Veblen and Enriques

- **Oswald VEBLEN (1880–1960)**

  - **in 1905 claimed to base Euclidean geometry on just two primitive concepts, *point* and *segment between*.**

  portrait

  But in 1907 Federigo ENRIQUES noted that

  - **Veblen's polar system, incompletely defined, was evidently a *third* primitive concept.**

  In 1911 Veblen did succeed with three primitives:

  - ***point*, *segment*, and *congruence of point pairs*.**

Oswald Veblen
(1880–1960)

Veblen and Enriques

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Pieri's *Point and Sphere*

♦ **Appeared in 1908** **first page** **Translated in Marchisotto & Smith 2007**

Pieri did base Euclidean geometry on just two primitive concepts:

point ***equidistance of a point O from two others*** **(*OP = OQ*)**

He used this train of definitions—*hold your hat*.

Pieri's axioms were frightfully complicated,

- **but we'd call them all *first-order*, except for continuity axioms.**

He *included all details* of his proofs!

Pieri 1908 received little acclaim,

- **except in Poland: Pieri 1915 is a translation. *Why???***

La Geometria Elementare istituita sulle nozioni di 'punto' e 'sfera'.

Memoria del prof. MARIO PIERI

(presentata dal Socio G. Castelnuovo, e approvata dal Socio G. Segre).

PREFAZIONE

« Die meisten Lehrbücher der Geometrie gehen zu « bald zum Besondern, nämlich zur Congruenz und Aen- « lichkeit der Dreiecke über; daher sie auch manche « begriffe nicht in der gehörigen Allgemeinheit auf- « stellen... Aehnliche Figuren sind nichts anderes als « homologe Stücke von ähnlichen räumlichen Systemen. « Der Betrachtung von centrischen Figuren sollte die « des centrischen Systems, und der Betrachtung von « symmetrischen Figuren die des symmetrischen Sy- « stems vorangeschickt werden. Uebrigens ist in vielen « Lehrbüchern, obgleich Natur und Kunst in allen ihren « Gebilden nach Symmetrie streben, dieser Begriff gar « nicht entwickelt. » G. C. von Staudt, '*Geom. d. Lage*', Vorwort (Nürnberg, 1847).

« Tutto sommato, parrebbe che gli elementi costrut- « tivi primordiali, che più spiccatamente intervengono « a formare lo spazio tattile-muscolare, non siano le « nozioni della retta e del piano, ma sì della 'distanza', « e quindi dei 'cerchi' e delle 'sfere' ». F. Enriques. *Problemi della scienza*, pag. 322 (Bologna, 1906).

In un lavoro del 1899 — dopo avere affermato « la possibilità di comporre tutta « quanta la Geometria elementare con queste due sole materie prime: il 'punto' « e una certa relazione fra tre punti a, b, c, che si può interpretar con le frasi « 'c dista da a quanto b', 'c appartiene alla sfera di b, centro a', 'le « coppie (a, b) ed (a, c) sono congrue fra loro', e rappresentar, se ci piace, « col simbolo '$c \varepsilon b_a$' » — soggiungevo: « l'eccessiva complicazione che involge « sinora la più gran parte d'un tal sistema (date le molte esigenze d'indole logico- « deduttiva, a cui si vuol sottostare) ne lascia tuttavia il desiderio, se non il bisogno, « di nuovi studî e di ricerche ulteriori (1) ». Frutto di così fatte ricerche il presente

(1) « *Della Geometria Elementare come sistema ipotetico-deduttivo* », nelle Memorie della Reale Accademia delle Scienze di Torino, v. $XLIX_2$ (1899), pag. 176.

Pieri 1908

Point and Sphere

Pieri's *Point and Sphere*

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Point & Sphere Definitions

- ***Q*** **is on the *sphere*** P_O **through** ***P*** **about** ***O*** $\Leftrightarrow$ $OP = OQ$

  if $O \neq Q$ **then** ***P*** **is *collinear* with** ***O,Q*** $\Leftrightarrow$ $P_O \cap P_Q = \{P\}$ **(Leibniz)**
  see figure 2

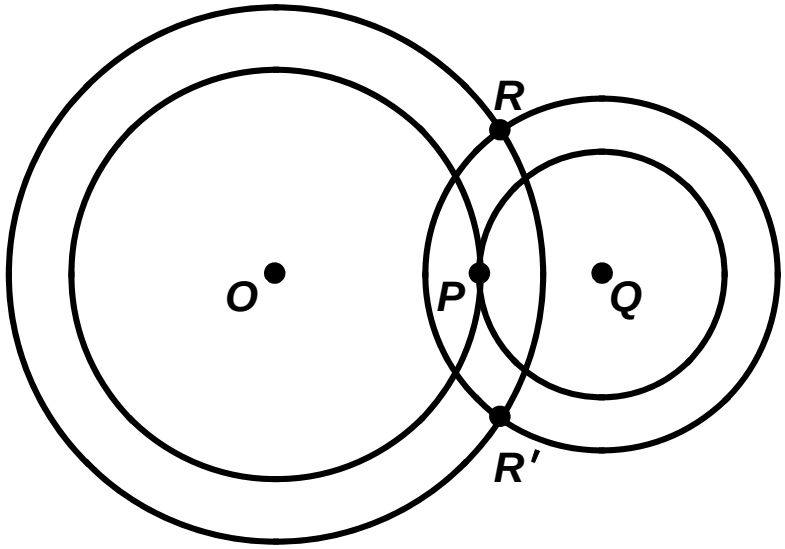
  $P = Q/O$, **the *reflection* of** ***Q*** **over** ***O*** $\Leftrightarrow$ ***O, P, Q*** **collinear &** $OP = OQ$

  spheres are *congruent* $\Leftrightarrow$ **they're related by reflection**

  $OP = QR$ **—point pairs are *congruent*—** $\Leftrightarrow$ ***R*** **is on a sphere about** ***Q*** **congruent to** P_O

  isometry **= transformation that preserves congruence of point pairs**

  (direct) *motion* = square of an isometry



P is collinear with O, Q; R is not.

Figure 2

Pieri's 1908 definition of *collinearity*

Point & Sphere Definitions

Q is on the *sphere* P_O through P about O ⇔ $OP = OQ$

if $O \neq Q$ then P is *collinear* with O,Q ⇔ $P_O \cap P_Q = \{P\}$ (Leibniz)
see figure 2

♦ $P = Q/O$, the *reflection* of Q over O ⇔ O, P, Q collinear & $OP = OQ$

spheres are *congruent* ⇔ they're related by reflection

$OP = QR$ —point pairs are *congruent*— ⇔ R is on a sphere about Q congruent to P_O

isometry = transformation that preserves congruence of point pairs

(direct) *motion* = square of an isometry

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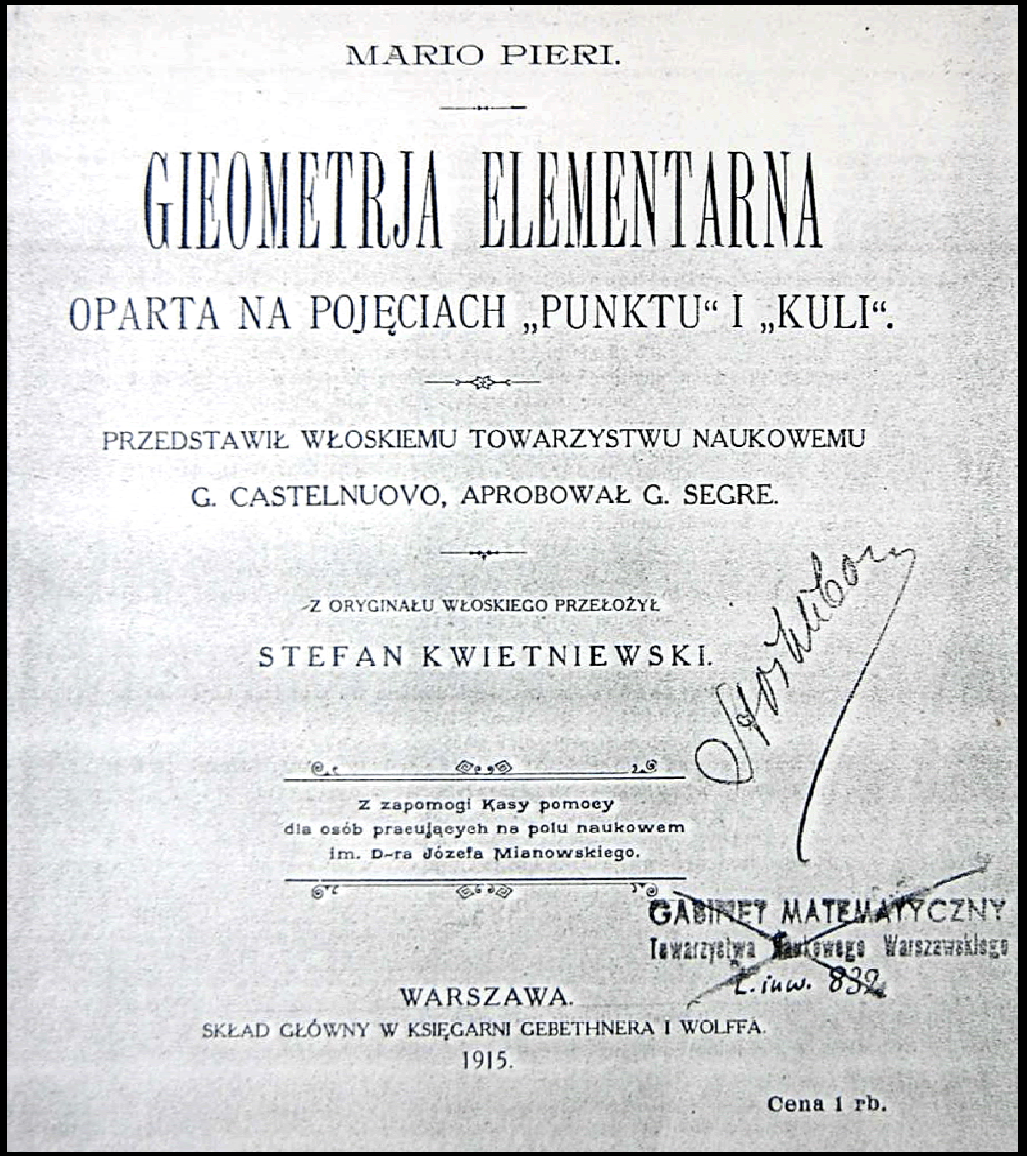

MARIO PIERI.

GIEOMETRJA ELEMENTARNA

OPARTA NA POJĘCIACH „PUNKTU“ I „KULI“.

PRZEDSTAWIŁ WŁOSKIEMU TOWARZYSTWU NAUKOWEMU

G. CASTELNUOVO, APROBOWAŁ G. SEGRE.

Z ORYGINAŁU WŁOSKIEGO PRZEŁOŻYŁ

STEFAN KWIETNIEWSKI.

Z zapomogi Kasy pomocy
dla osób pracujących na polu naukowem
im. D-ra Józefa Mianowskiego.

WARSZAWA.
SKŁAD GŁÓWNY W KSIĘGARNI GEBETHNERA I WOLFFA.
1915.

Cena 1 rb.

Pieri 1915

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Tarski

- **Early career**

For research and for his university course Tarski adopted the *hypothetical-deductive system* framework.

In the late 1920s he was beginning to use *first-order* logic, which avoided use of sets.

Pieri's 1908 *Point and Sphere* system, which fit that framework, was adapted for Tarski's system of geometry.

Tarski's Early Career

Alfred TARSKI (1901-1983)

- **earned the doctorate in logic at Warsaw under Leśniewski,**
- **but the economy and antisemitism made it hard to find a job.**

In 1927 Tarski was a high-school teacher and university assistant,

- **starting on the path to become the world's top logician.**

His research featured application of logical techniques to geometry.

portrait

Alfred Tarski
(1901–1983)

Little is known about Tarski's pre-1939 life in Poland. Any information, particularly from others' correspondence, would be of great value!

Tarski

Early career

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  In the late 1920s he was beginning to use *first-order* logic, which avoided use of sets.

  For his system of geometry, Tarski adapted Pieri's 1908 *Point and Sphere*, which fit that framework.

Tarski's System of Geometry

- **Tarski's primitive concepts were *point* and *two* relations,**
  - ***congruence* of point pairs**
  - ***betweenness* of triples.**

  Thus he was also following Veblen's 1911 work.

  Tarski's axioms were *vastly simpler* than Pieri's,
  - **and his system enabled much *deeper* research.**
  - **Its models are precisely the coordinate geometries over real-closed ordered fields.**

  He didn't publish until *1959*! **title page**

  The proofs finally appeared in Schwabhäuser, Szmielew & Tarski 1983.

THE AXIOMATIC METHOD

WITH SPECIAL REFERENCE TO GEOMETRY AND PHYSICS

Proceedings of an International Symposium held at the University of California, Berkeley, December 26, 1957 — January 4, 1958

Edited by

LEON HENKIN
Professor of Mathematics, University of California, Berkeley

PATRICK SUPPES
Associate Professor of Philosophy, Stanford University

ALFRED TARSKI
Professor of Mathematics and Research Professor, University of California, Berkeley

1959

NORTH-HOLLAND PUBLISHING COMPANY
AMSTERDAM

WHAT IS ELEMENTARY GEOMETRY?

ALFRED TARSKI
Institute for Basic Research in Science, University of California, Berkeley, California, U.S.A.

In colloquial language the term *elementary geometry* is used loosely to refer to the body of notions and theorems which, following the tradition of Euclid's *Elements*, form the subject matter of geometry courses in secondary schools. Thus the term has no well determined meaning and can be subjected to various interpretations. If we wish to make elementary geometry a topic of metamathematical investigation and to obtain exact results (not within, but) about this discipline, then a choice of a definite interpretation becomes necessary. In fact, we have then to describe precisely which sentences can be formulated in elementary geometry and which among them can be recognized as valid; in other words, we have to determine the means of expression and proof with which the discipline is provided.

In this paper we shall primarily concern ourselves with a conception of elementary geometry which can roughly be described as follows: *we regard as elementary that part of Euclidean geometry which can be formulated and established without the help of any set-theoretical devices.* [1]

More precisely, elementary geometry is conceived here as a theory with standard formalization in the sense of [9]. [2] It is formalized within ele-

[1] The paper was prepared for publication while the author was working on a research project in the foundations of mathematics sponsored by the U.S. National Science Foundation.

[2] One of the main purposes of this paper is to exhibit the significance of notions and methods of modern logic and metamathematics for the study of the foundations of geometry. For logical and metamathematical notions involved in the discussion consult [8] and [9] (see the bibliography at the end of the paper). The main metamathematical result upon which the discussion is based was established in [7]. For algebraic notions and results consult [11].

Several articles in this volume are related to the present paper in methods and results. This applies in the first place to Scott [5] and Szmielew [6], and to some extent also to Robinson [3].

Tarski [1957] 1959

Tarski's System of Geometry

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Tarski's axioms were *vastly simpler* than Pieri's,

- **and his system enabled much *deeper* research.**
- **Its models are precisely the coordinate geometries over real-closed ordered fields.**

He didn't publish until *1959*! **title page**

♦ **The proofs finally appeared in Schwabhäuser, Szmielew & Tarski 1983.**

Definitions and Nondefinability in Geometry

James T. Smith

- Pasch
- Peano
- Pieri
- Veblen & Enriques
- Pieri's *Point & Sphere*
- Tarski
- ♦ Nondefinability

Nondefinability

♦ **Consider geometry based on *congruence* D of point pairs and *betweenness* B of triples.**

Pieri, in 1908, presented a formula $Fabc$ involving just D, and proved

- $(\forall a, b, c)\ [\ Babc \Leftrightarrow Fabc\ ]$ — **B is definable from D.**

Veblen, in 1904, claimed the reverse: D definable from B.
In 1907, Enriques objected.

Tarski *refuted* Veblen easily in 1935.

Adolf LINDENBAUM and Tarski, by a similar 1935 argument, proved Pieri *optimal*:
no formula F can be a definition of B or D if F is constructed just from *binary* relations R that are definable in Tarski's system.

This work has led to related studies.

Refuting Veblen

This required a precise definition of *definition*.

That's possible with Tarski's geometry.

Suppose *Fabcd* were a formula involving just *B* for which

- **$(\forall a, b, c, d)$ [*Dabcd* $\Leftrightarrow$ *Fabcd*] .**

There's an affine transformation *A* that's not a similarity.

- ***F* would be invariant under *A*, because *B* is.**

That's contradictory, because *D* is not.

Thus no such *F* exists: *D* is *not* definable from *B* alone.

This method of argument is due to Padoa in 1900.

But the result is by Tarski in 1935.

It required logical methods unfamiliar until then.

Nondefinability

Consider geometry based on *congruence* D of point pairs and *betweenness* B of triples.

Pieri, in 1908, presented a formula $Fabc$ involving just D, and proved

- **$(\forall a, b, c)\ [\ Babc \Leftrightarrow Fabc\ ]$ — B is definable from D.**

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♦ **This work has led to related studies.**

Related Studies

Many related studies are reported in Schwabhäuser, Szmielew & Tarski 1983.

In 1990–1991 Victor PAMBUCCIAN showed that Pieri's result can be improved, if the logic is strengthened:

Let y, z be an arbitrary, fixed, pair of $\neq$ points.
Then D, and hence B, can be defined in terms of

- **the *single binary* relation $Uab \Leftrightarrow Dabyz$,**

provided the language and logic permit *countably infinite conjunctions*.

His argument is what Pieri used in 1908 to analyze the statement,

- **distance$(a,b) = (m/2^n)$ distance(y,z) !**

Canadian Society for History and Philosophy of Mathematics
Société canadienne d'histoire et de philosophie des mathématiques

Annual Meeting, 30 May 2010

Thank you for your interest!

James T. Smith, Professor Emeritus
San Francisco State University

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Definitions and Nondefinability in Geometry: Legacies of Mario Pieri and Alfred Tarski

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