CUBIC AND QUARTIC FORMULAS

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Quadratic formula

You've met the *quadratic formula* in algebra courses: the solution of the quadratic equation

$$ax^2 + bx + c = 0$$

with specified real coefficients $a \neq 0$, b, and c is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} .$$

You can derive the formula as follows. First, divide the quadratic by a to get the equivalent equation

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 .$$

Now substitute $x = y + d$. You'll choose d later so that the resulting equation is easy to solve. Making the substitution, you get

$$(y + d)^2 + \frac{b}{a}(y + d) + \frac{c}{a} = 0 .$$

Work this out, ignoring some details that won't be necessary:

$$y^2 + 2dy + d^2 \frac{b}{a} y + \text{constants} = 0$$

$$y^2 + \left(2d + \frac{b}{a}\right) y + \text{constants} = 0.$$

If the y coefficient were zero, you could move the constants to the other side and solve for y by taking the square root. Thus you can find y easily if you let $d = -b/(2a)$. Do that and work out the details: the last equation displayed above becomes

$$y^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$y^2 = \frac{b^2 - 4ac}{4a^2}\,.$$

The quantity $D = b^2 - 4ac$ is called the *discriminant* of the quadratic: you can write $y = \pm\sqrt{D}\,/(2a)$. Finally, the desired solution is

$$x = d + y = -\frac{b}{2a} \pm \frac{\sqrt{D}}{2a}$$

—the Quadratic Formula. If $D < 0$ then you can write x in complex form:

$$x = d + y = -\frac{b}{2a} \pm \frac{\sqrt{-D}}{2a}\,i\,.$$

Cubic formula

It's possible to imitate this process to derive a formula for solving a *cubic* equation

$$ax^3 + bx^2 + cx + d = 0$$

with specified real coefficients $a \neq 0$, b, c, and d. First, divide the cubic by a to get the equivalent equation

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0\,.$$

Next, make a substitution $x = y + e$, and choose e so that the resulting equation is easier to solve. Experience with the quadratic equation suggests that you might be able to make one term of the cubic disappear, so that the resulting equation is like one of these:

$$y^3 + fy^2 + g = 0 \qquad\qquad y^3 + py + q = 0.$$

Experimentation would show that you can reach the form on the left, but it doesn't lead anywhere. However, you can also get the one on the right, and it *does* help. In fact, if you let $e = -b/(3a)$ —that is, substitute $x = y - b/(3a)$ —then you get, after considerable algebraic labor, the equation

$$y^3 + py + q = 0,$$

where

$$p = \frac{-b^2}{3a^2} + \frac{c}{a} \qquad\qquad q = \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a}\,.$$

You've now reduced the problem of solving the original cubic to one of solving $y^3 + py + q = 0$ and setting $x = y - b/(3a)$.

After centuries' experience, mathematicians found a trick to solve this simpler cubic. Set $y = z - p/(3z)$ and consider the resulting equation. After considerable algebraic labor, it comes out to

$$z^6 + qz^3 - \frac{p^3}{27} = 0.$$

This equation is quadratic in z^3. You can find z^3 by using the quadratic formula:

$$z^3 = -\frac{q}{2} \pm \sqrt{D},$$

where

$$D = \frac{q^2}{4} + \frac{p^3}{27}.$$

This D is called the *discriminant* of the cubic equation. Taking the cube root, you get

$$\sqrt[3]{-\frac{q}{2} \pm \sqrt{D}}.$$

Together with the equations

$$y = z - \frac{p}{3z} \qquad\qquad x = y - \frac{b}{3a}$$

this is called the *cubic* formula: it shows you how to compute the solution x of the original cubic equation. Actually, the equation for z gives three complex cube roots for each of the $+$ and $-$ signs, hence six different formulas for z. But when you substitute these in the equation for y, at most three different y values will result, and the last equation will thus give at most three distince roots x.

Look at the cubic formula in more detail. When $D \geq 0$, you can select one of the two real square roots $\pm\sqrt{D}$, then find three cube roots $z = z_0$, z_1, and z_2 of $-q/2 \pm\sqrt{D}$ as follows. Let z_0 be the real cube root, then $z_1 = \omega z_0$ and $z_2 = \omega^2 z_0$, where ω and ω^2 are the two complex cube roots of 1:

$$\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \qquad\qquad \omega^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

You can use simple algebra of complex numbers to find the corresponding values y and x.

If you get $z = 0$ in this process, then $(q/2)^2 = D$, hence $p = 0$. In that case, you're just solving the equation $y^3 + q = 0$, which has one root $y = 0$ if $q = 0$, and three roots $y = -q^{1/3}$ otherwise. In the former case, $D = p = q = 0$. If $D = 0$ but $p \neq 0$, then two of the y values coincide, and the equation has distinct single and double real roots.

The situation is more interesting when $D < 0$, for then $\sqrt{D}$ is imaginary, and you have to find the cube roots z of the complex number

$$A = -\frac{q}{2} + \sqrt{-D}\, i\,.$$

In this case,

$$0 > D = \frac{q^2}{4} + \frac{p^3}{27}\,,$$

so $p < 0$. Next,

$$|A|^2 = \left(-\frac{q}{2}\right)^2 + \left(\sqrt{-D}\right)^2 = \frac{q^2}{4} - D = -\frac{p^3}{27}$$

$$|A| = \left(\frac{|p|}{3}\right)^{\frac{3}{2}}.$$

Now you can write $A = |A| \operatorname{cis}\theta$, where

$$\cos\theta = \frac{\operatorname{Re} A}{|A|} = \frac{-\frac{q}{2}}{\left(\frac{|p|}{3}\right)^{\frac{3}{2}}} = \frac{3\sqrt{3}\,q}{2p\sqrt{-p}}\,.$$

Given coefficients a to d, you can compute p and q from the previous equations, calculate $\cos\theta$, then determine θ. Now let

$$r = \sqrt[3]{|A|} = \sqrt{\frac{|p|}{3}}\,,$$

so that De Moivre's formula yields the three values

$$z = \sqrt[3]{A} = r \operatorname{cis}\psi,$$

where

$$\psi = \frac{\theta}{3},\ \frac{\theta}{3} + 120^\circ,\ \frac{\theta}{3} + 240^\circ.$$

The details of the final calculation of y and x are interesting, too. You'll get

$$
\begin{aligned}
y &= z - \frac{p}{3z} = r\,\mathrm{cis}\,\psi - \frac{p}{3r\,\mathrm{cis}\,\psi} \\
&= r\,\mathrm{cis}\,\psi - \frac{p}{3r}(\mathrm{cis}\,\psi)^{-1} = r\,\mathrm{cis}\,\psi - \frac{p}{3r}\,\mathrm{cis}(-\psi) \\
&= r[\cos\psi + i\sin\psi] - \frac{p}{3r}[\cos(-\psi) + i\sin(-\psi)] \\
&= \left(r - \frac{p}{3r}\right)\cos\psi + \left(r + \frac{p}{3r}\right)i\sin\psi\,.
\end{aligned}
$$

But

$$
r + \frac{p}{3r} = \sqrt{\frac{|p|}{3}} + \frac{-|p|}{\sqrt[3]{\frac{|p|}{3}}} = 0,
$$

hence all three y values are real! You have

$$
y = \left(r - \frac{p}{3r}\right)\cos\psi
$$

for the three values of ψ given earlier. In this case, where $D < 0$, you've found the real roots of a real cubic equation through use of complex numbers and trigonometry. It's now known that some such 'detour' through the complex numbers is *necessary* to find a formula for these roots!

Example cubics

1. To solve $ax^3 + bx^2 + cx + d = 0$ with $a,b,c,d = 4,3,2,1$ follow the derivation of the cubic formula, setting

$$
x = y - \frac{b}{3a} = y - \frac{1}{4}
$$

$$
p = \frac{-b^2}{3a^2} + \frac{c}{a} = \frac{5}{16}
$$

$$q = \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a} = \frac{5}{32}$$

$$y = z - \frac{p}{3z} = z - \frac{5}{48z}$$

$$x = y - \frac{b}{3a} = y - \frac{1}{4}$$

$$D = \frac{q^2}{4} + \frac{p^3}{27} = \frac{25}{3456}\,.$$

Since $D > 0$, there are one real and two conjugate complex roots. Compute the real root as follows:

$$z = \sqrt[3]{-\frac{q}{2} \pm \sqrt{D}} = \sqrt[3]{\frac{-45 \pm 20\sqrt{6}}{576}} \approx \begin{cases} 0.1906230 \\ -0.5464530 \end{cases}.$$

Both z values give the same value of

$$y = z - \frac{5}{48z} \approx -0.35583$$

$$x = y - \frac{1}{4} \approx -0.60583.$$

Substituting this x value in the left-hand side of the original equation yields a value of about -1×10^{-6} —acceptable accuracy. The complex roots are computed from the other two cube roots:

$$z \approx 0.1906280\left(-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right) \approx -0.0953115 \pm 0.1650844i$$

$$y = z - \frac{5}{48z} \approx 0.17792 \pm 0.63827i$$

$$x = y - \frac{1}{4} \approx -0.07208 \pm 0.63827i.$$

Substituting these x values in the left-hand side of the original equation yields values with norms of about $2 \times 10^{-4}i$ —acceptable accuracy.

2. To solve $y^3 + py + q = 0$ with $p, q = -2, 1$ follow the derivation of the cubic formula, to get

$$D = \frac{q^2}{4} + \frac{p^3}{27} = -\frac{5}{108}\,.$$

Since $D < 0$, there are three real roots, determined as follows:

$$\cos\theta = \frac{3\sqrt{3}\,q}{2p\sqrt{-p}} \approx -0.91856$$

$$\theta \approx 156.716^\circ$$

$$\psi = \frac{\theta}{3},\ \frac{\theta}{3} + 120^\circ,\ \frac{\theta}{3} + 240^\circ \approx 52.239^\circ,\ 172.239^\circ,\ 292.239^\circ$$

$$r = \sqrt{\frac{|p|}{3}} = \sqrt{\frac{2}{3}}$$

$$y = \left(r - \frac{p}{3r}\right)\cos\psi \approx 1.00000,\ -1.61803,\ 0.61803$$

The first value of y suggests that $y = 1$ is an exact root of the cubic. That's easy to verify: in fact,

$$y^3 - 2y + 1 = (y-1)(y^2 + y - 1)\,.$$

The roots of the right-hand factor are $(-1 \pm \sqrt{5})/2$, which agree with the other two computed y values.

Quartic formula

Why stop with cubics? Why not apply the same method to a *quartic* equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

with real coefficients $a \neq 0$, b, c, d, and e? First divide the quartic by a to obtain the equivalent equation

$$x^4 + \frac{b}{a}x^3 + \frac{c}{a}x^2 + \frac{d}{a}x + \frac{e}{a} = 0\,.$$

The substitutions $x = y - b/(2a)$ and $x = y - b/(3a)$ worked for the quadratic and cubic; try $x = y - b/(4a)$ for the quartic. After a lot of algebra, you'll get an equation of the form

$$y^4 + py^2 + qy + r = 0$$

for certain values p, q, and r that you can compute from the original coefficients. Not long after the trick was discovered that led to the solution of the cubic, mathematicians discovered one that leads to the solution of this quartic. Write it as

$$y^4 = -py^2 - qy - r.$$

Now manipulate this equation, using a value z that will be determined later:

$$y^4 + 2y^2z^2 + z^4 = -py^2 - qy - r + 2y^2z^2 + z^4$$
$$(y^2 + z^2)^2 = (2z^2 - p)y^2 - qy + (z^4 - r).$$

Later, you'll determine z so that the right-hand side of this last equation is $(fy + g)^2$ for some particular values f and g. Then you'll have

$$(y^2 + z^2)^2 = (fy + g)^2$$
$$y^2 + z^2 = \pm(fy + g)$$
$$\begin{cases} y^2 - fy + (z^2 - g) = 0, \text{ or} \\ y^2 + fy + (z^2 + g) = 0. \end{cases}$$

Once z then f and g are found, you can solve for y by using the quadratic formula on the last pair of equations.

Thus you have to find z, f, and g so that

$$(2z^2 - p)y^2 - qy + (z^4 - r) = (fy + g)^2.$$

Consider the equations formed by setting each side of this equal to zero. The right-hand equation $(fy + g)^2 = 0$ would have just one root $y = -g/f$; thus the left-hand equation

$$(2z^2 - p)y^2 - qy + (z^4 - r) = 0$$

would have just one root also. But you can solve this equation for y by using the quadratic formula, and for that formula to yield just one root its discriminant must be zero. That is, you can find z, f, and g just when

$$q^2 - 4(2z^2 - p)(z^4 - r) = 0$$
$$-8z^6 + 4pz^4 + 8rz^2 + (q^2 - 4pr) = 0$$
$$\begin{cases} w = z^2 \\ 8w^3 - 4pw^2 - 8rw + (4pr - q^2) = 0. \end{cases}$$

With sufficient labor, you can solve the last equation by using the cubic formula, getting at most three solutions $w = w_0,\ w_1,$ and w_2. For each j, the equation

$$(2w_j - p)y^2 - qy + (w_j^2 - r) = 0$$

has a single root

$$y = \frac{q}{2(2w_j - p)}.$$

Therefore,

$$(2w_j - p)y^2 - qy + (w_j^2 - r) = (2w_j - p)\left(y - \frac{q}{2(2w_j - p)} \right)^2$$

$$= \left(y\sqrt{2w_j - p} - \frac{q}{2\sqrt{2w_j - p}} \right)^2 = (fy + g)^2,$$

where

$$f = \sqrt{2w_j - p} \qquad\qquad g = -\frac{q}{2f}.$$

With these three values of f and g, solve the two quadratic equations

$$\begin{cases} y^2 - fy + (z^2 - g) = 0 \\ y^2 + fy + (z^2 + g) = 0 \end{cases}$$

for y. This yields as many as twelve possible y formulas, but at most four can have distinct values. Finally, calculate the roots $x = y - b/(4a)$.

Example quartic

To solve $ax^4 + bx^3 + cx^2 + dx + e = 0$ with $a,b,c,d,e = 2,2,-3,-3,-4,$ follow the quartic formula, dividing by a and setting

$$x = y - \frac{b}{3a} = y - \frac{1}{4},$$

to get the equation $y^4 + py^2 + qy + r = 0$, where $p = -15/8$, $q = -5/8$, and $r = -443/256$. Next, solve the equation $8w^3 - 4pw^2 - 8rw + (4pr - q^2) = 0$ by the cubic formula, obtaining three solutions

$$w_0 \approx -0.918531$$
$$w_1 \approx -0.00948425 + 1.30880i$$
$$w_2 = \overline{w_1}.$$

With $j = 0$ compute

$$f = \sqrt{2w_j - p}\,, \qquad g = -\frac{q}{2f}\,,$$

solve $y^2 - fy + (w_j - g) = 0$ for two values $y = y_0$ and y_1, and set $x_k = y_k - b/(4a)$ for $k = 0,1$ to get

$$x_0 \approx -1.74398, \qquad x_1 \approx 1.43875.$$

Next, solve $y^2 + fy + (w_j + g) = 0$ for two values $y = y_2$ and y_3, and set $x_k = y_k - b/(4a)$ for $k = 2,3$ to get

$$x_2 \approx -0.347387 + 0.822439i, \qquad x_3 = \overline{x_2}.$$

With $j = 1,2$ this process yields the same four roots x. Substituting these x values back in the original quartic equation yields values with norm less than 2×10^{-10} —acceptable accuracy. Like the cubic examples considered earlier, these roots can be expressed exactly with radicals and trigonometric functions. However, their expressions would be so complicated that checking and using them would be impractical.

History

In 1505 Scipione de Ferro (1465–1526), Professor of Mathematics at Bologna, discovered a method for solving certain cubic equations, but didn't publicize it beyond his students, preferring to use it secretly to establish himself as a problem solver. In 1535 one of the students, Antonio Maria Fior, challenged Nicolo Fontano of Brescia, nicknamed Tartaglia (stammerer), and the latter discovered the same method. Tartaglia was a self-taught mathematics teacher who had already written the first serious treatise on ballistics, and would later translate Euclid into Italian. He was pressured to reveal the secret by Geronimo Cardano (1501–1575), a famous and infamous professor of mathematics, medicine, and roguery at Pavia and other North Italian universitites. On promise of secrecy, Tartaglia showed him the method. But Cardano set to work elaborating it, and soon his student Lodovico Ferrari had extended it to solve quartics. Cardano published both in his *Ars magna* in 1545. This is one of several of his treatises typical of the time: encyclopedias of everything, from occult descriptions of demons to natural history to theoretical mathematics. Tartaglia became involved with Cardano and Ferrari in a public priority dispute, and faded from the scene. Cardano became famous for studies of syphilis and typhus, and for an autobiography (of a rogue and scoundrel). He wrote one of the first books on probability, published posthumously. Cardano was imprisoned in 1570 for the heresy of casting Christ's horoscope; but the Pope rethought the matter, released him, then hired him as papal astrologer!

The Italians were handicapped by their lack of notation for variables. This was provided by François Viète (1540–1603), a politician and lawyer from Brittany involved with the Huguenot cause. He pursued mathematics as a hobby, especially while out of office, and published privately a number of treatises in which he used an algebraic symbolism similar to ours. The treatment of cubic and quartic equations given here is essentially his.

The modern theory of roots of polynomials in general was not developed until the middle 1600s by Descartes and others. The first complete treatment of cubic and quartic equations was given by Euler in 1732. During the 1600s and 1700s, mathematicians regarded extension of these methods to quintic and higher equations as a major open problem, but met with no success. Not until 1800 did Gauss prove that *every polynomial has at least one complex root*, and not until about 1830 did Galois, Abel, and others show that roots of quintic and higher degree equations could not, in general, be found by the familiar methods involving algebraic operations and extraction of roots.

Exercises

Find all roots of each of the following cubics. Verify each real root by substituting it for x and calculating the left hand side of the equation.

1. $x^3 - 6x - 6 = 0$
2. $3x^3 - 6x^2 - 2 = 0$
3. $x^3 - 6x + 2 = 0$
4. $x^3 + x^2 - 2x - 1 = 0$

Acknowledgement

Thanks to Singapore student Jessica Ng for finding two serious typographical errors in the previous version of this note.

Solutions

1. To solve $x^3 + px + q = 0$ with $p = q = -6$, set $x = y = z - p/(3z) = z + 2/z$. The equation then becomes

$$z^3 + \frac{8}{z^3} - 6 = 0$$
$$z^6 - 6z^3 + 8 = 0$$
$$(z^3 - 4)(z^3 - 2) = 0,$$

hence $z^3 = 4$ or $z^3 = 2$. Select $z^3 = 4$ (it makes no difference which alternative you pick). Then $z = z_0,\ z_1,$ or $z_2,$ where

$$z_0 = \sqrt[3]{4} \qquad z_1 = z_1\omega \qquad z_2 = z_0\bar{\omega},$$

and $x = x_0,\ x_1,$ or $x_2,$ where

$$x_0 = z_0 + \frac{2}{z_0} = \sqrt[3]{4} + \sqrt[6]{4} \approx 2.84732$$

$$x_1 = z_1 + \frac{2}{z_1} = z_0\omega + \frac{2}{z_0}\bar{\omega} = z_0\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + \frac{2}{z_0}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

$$= -\frac{1}{2}\left(z_0 + \frac{2}{z_0}\right) + \frac{\sqrt{3}}{2}\left(z_0 - \frac{2}{z_0}\right)i = \frac{x_0}{2} + \frac{\sqrt{3}}{2}\left(\sqrt[3]{4} - \sqrt[6]{4}\right)i$$

$$\approx -1.42366 - 0.28361i$$

$$x_2 = \overline{x_1} \approx -1.42366 + 0.28361i.$$

2. To solve $3x^3 - 6x^2 - 2 = 0$, set $x = y - (-6)/(3\cdot 3) = y + 2/3$. The equation becomes $y^3 - (4/3)\,y - 34/27 = 0$. Now set $y = z + (4/3)/(3z) = z + (4/9)z$, and the equation becomes $z^3 + (64/729)z^3 - 34/27 = 0$, i.e. $729z^6 - 918z^3 + 64 = 0$. By the quadratic formula, $z^3 = 32/27$ or $2/27$. Take the former—it makes no difference which. Then $z = z_0,\ z_1,$ or $z_2,$ where

$$z_0 = \frac{\sqrt[3]{2}}{3} \qquad z_1 = z_1\omega \qquad z_2 = z_0\bar{\omega},$$

and $x = x_j = y_j + 2/3$,

$$y_0 = z_0 + \frac{4}{9z_0} = \frac{2^{1/3} + 2^{5/3}}{3}$$

$$y_1 = z_0\omega + \frac{4}{9z_0}\bar{\omega} = -\frac{1}{2}y_0 + \frac{\sqrt{3}}{6}\left(2^{1/3} - 2^{5/3}\right)i$$
$$y_2 = \overline{y_1}.$$

Thus

$$x_0 \approx 2.14490$$
$$x_1 \approx -0.072452 - 0.55278i$$
$$x_2 \approx -0.072452 + 0.55278i.$$

3. To solve $x^3 + px + q = 0$, where $p = -6$ and $q = 2$, set $x = z - p/(3z) = z + 2/z$. The equation becomes

$$z^3 + \frac{8}{z^3} + 2 = 0$$

$$z^6 + 2z^3 + 8 = 0$$

$$z^3 = -1 \pm \sqrt{7}\, i = r\,\text{cis}\,\theta$$

$$r = \sqrt{1+7} = 2\sqrt{2}$$

$$\cos\theta = -\frac{1}{r} = -\frac{\sqrt{2}}{4}$$

$$\theta = \theta_0,\ \theta_0 + 2\pi,\ \text{or}\ \theta_0 + 4\pi$$

$$\theta_0 = \cos^{-1}\left(-\frac{\sqrt{2}}{4}\right) \approx 1.93216$$

$$z = \sqrt[3]{r}\ \text{cis}\,\frac{\theta}{3} = \sqrt{2}\ \text{cis}\,\frac{\theta}{3}$$

$$x = z + \frac{2}{z} = \sqrt{2}\ \text{cis}\,\frac{\theta}{3} + \frac{2}{\sqrt{2}\ \text{cis}\,\frac{\theta}{3}}$$

$$= \sqrt{2}\ \text{cis}\,\frac{\theta}{3} + \sqrt{2}\ \text{cis}\left(-\frac{\theta}{3}\right) = 2\sqrt{2}\cos\frac{\theta}{3}$$

$$\approx 2.2618052,\ -2.6016777,\ \text{or}\ 0.33987722.$$

4. To solve $x^3 + x^2 - 2x - 1 = 0$, set $x = y - 1/3$. The equation becomes $y^3 - (7/3)y - 7/27 = 0$. Now set $y = z + 7/(9z)$, and the equation becomes

$$z^3 + \frac{343}{729z^3} - \frac{7}{27} = 0$$

$$z^6 - \frac{7}{27}z^3 + \frac{343}{729} = 0$$

$$z^3 = \frac{7}{54} \pm \frac{21\sqrt{3}}{54}i = r\operatorname{cis}\theta$$

$$r = \frac{7^{3/2}}{3^3}$$

$$\sqrt[3]{r} = \frac{\sqrt{7}}{3}$$

$$\cos\theta = \frac{\sqrt{7}}{14}$$

$$\theta = \theta_0,\ \theta_0 + 2\pi,\ \text{or}\ \theta_0 + 4\pi$$

$$\theta_0 = \cos^{-1}\frac{\sqrt{7}}{14} \approx 1.38067$$

$$z = \sqrt[3]{r}\operatorname{cis}\frac{\theta}{3} = \frac{\sqrt{7}}{3}\operatorname{cis}\frac{\theta}{3}$$

$$y = z + \frac{7}{9z} = \frac{\sqrt{7}}{3}\operatorname{cis}\frac{\theta}{3} + \frac{7}{3\sqrt{7}\operatorname{cis}\frac{\theta}{3}}$$

$$= \frac{\sqrt{7}}{3}\operatorname{cis}\frac{\theta}{3} + \frac{\sqrt{7}}{3}\operatorname{cis}\left(-\frac{\theta}{3}\right) = \frac{2\sqrt{7}}{3}\cos\frac{\theta}{3}$$

$$x = y - \frac{1}{3} \approx 1.2469792,\ -1.8019369,\ \text{or}\ -0.4450425.$$