SONDERDRUCK aus

# Grundlagen der Geometrie und algebraische Methoden

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## FOUNDATIONS OF METRIC GEOMETRY OF ARBITRARY DIMENSION

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### I. Introduction

"Metric geometries" discussed by various authors have the following characteristics. They satisfy some simple incidence axioms and orthogonality axioms. They admit all possible reflectional symmetries and satisfy the three - reflections axioms. They can be embedded in projective "ideal geometries", which can then be coordinatized by commutative fields of scalars whose characteristics are not 2. Their orthogonality relations are representable by symmetric bilinear forms, and the groups of their motions - compositions of reflections - are isomorphic to subgroups of the projektive orthogonal groups. Metric geometries include the classical Euclidean, elliptic, and Bolyai-Lobatschevsky geometries.

In this lecture we consider metric geometries without regard to dimension.

There are four different approaches. The most basic is the "analytic" approach suggested above. The "foundation problem" consists in relating other approaches to this one. The oldest approach is of course the "synthetic": one starts with axioms about points, lines, planes, orthogonality, and so on. Another approach is the "group theoretic": here we axiomatize the notion of the group of motions of a metric geometry. The last approach is the "lattice theoretic": we characterize in an algebraically convenient way the family of all flat subspaces of a metric geometry. (There is an analogous list of approaches to projective geometry: BAER [3] and FRINK [5] have given equivalent analytic, synthetic, and lattice theoretic presentations.)

Equivalent synthetic and group theoretic developments of plane metric geometry were given by BACHMANN [2] in his book AGS. Equivalent presentations in three dimensions were achieved by his students SCHERF [11] and AHRENS [1]. The author [12] has presented a synthetic axiom system for metric geometry of arbitrary (perhaps infinite) dimension that in the finite dimensional case is equivalent to the group theoretic approach of KINDER [6]. EWALD [4] has announced an equivalent group theoretic approach to metric geometry of arbitrary (perhaps infinite) dimension.

For finite dimensional elliptic geometry, KINDER & WOLFF [7] and KINDER [8] have given equivalent lattice theoretic and group theoretic presentations. The author announces here his extension of their work to arbitrary (perhaps infinite) dimension.

Part of the author's work has already appeared [13], [14]. The remainder of this lecture is a summary of work that is yet unpublished [15], [16], [17].

## II. Generalized metric geometries

Consider an orthogonal geometry $(r, G, E, \perp)$ - here r is a nonempty set called the entire space; singleton subsets of r are called points; G and E are families of subsets of r called lines and planes; $\perp$ is a binary relation on G called orthogonality; and some familiar incidence and orthogonality axioms are satisfied - see [13], [14]. A reflection in a flat x is a self inverse permutation of r that preserves collinearity and orthogonality, and leaves fixed all flats incident with x, but pointwise fixed only those in x. If $(r, G, E, \perp)$ satisfies the following Symmetry Axioms, then it is called a generalized metric geometry.

Axiom S1. Each point has a reflection.
Axiom S2. Each line has a reflection.
Axiom S2 is redundant in case the geometry is elliptic.

By reducing spatial questions to questions about planes, then using results in AGS, we can prove from these axioms that a nonempty flat x has a reflection if and only if it is orthocomplemented - that is, the entire space is the join of x and its complement [o,x] at some point o in x. Moreover, in this case the reflection is unique; we denote it by $\sigma_x$. For arbitrary points o



and p and flats x and y, we can prove

$$\sigma_o = \sigma_p \longleftrightarrow o = p$$
$$\sigma_o \mid \sigma_p \longleftrightarrow o \sim p$$
$$\sigma_x = \sigma_y \longleftrightarrow x = y \text{ or } x = y^{\perp}$$
$$\sigma_x \mid \sigma_y \longleftrightarrow x \subseteq y \text{ or } x \supseteq y \text{ or } x \subseteq y^{\perp} \text{ or } x \supseteq y^{\perp} \text{ or } x \perp y \text{ or } \sigma_x(y) = y^{\perp}$$
$$x \perp y \longrightarrow \sigma_x \sigma_y = \sigma_{x \cap y} \quad \sigma_{x \vee y}$$
$$o \subseteq x \longrightarrow \sigma_o \sigma_x = \sigma_{[o,x]}.$$

A motion of $(r, G, E, \perp)$ is a composition of point and line reflections. Is this really the right definition? It excludes reflections in orthocomplemented infinite dimensional infinite codimensional flats.

## III. Metric geometries

A metric geometry is a generalized metric geometry that satisfies the following Three-Reflections Axioms:

Axiom 3R1. The composition of reflections in three points in a line g is a reflection in a fourth point in g.

Axiom 3R2. The compositions of the reflections in three lines through a point o in a plane e is a reflection in a fourth line through o in e.

Axiom 3R2 is redundant is case the geometry is elliptic.

Using methods of AHRENS [1] and results in AGS we can derive the following theorems. First, the Vierkanthypothese for the incidence geometry (r, G, E) and Desargues' Theorem for the projective geometry of all flats through a given point. From these it follows that our geometry can be embedded in a projective "ideal geometry" - see WYLER [18] - which can be coordinatized by a division ring of scalars - see BAER [3] - and that the orthogonality relation is represented by an Hermitian form - see LENZ [10] and SMITH [14].

Next, Pappus' and Fano's Theorems for the ideal geometry. From these it follows that the scalars form a commutative field whose characteristic is not two. Last, the Höhenschnittpunktsatz, from which it follows that the Hermitian form is bilinear.

The group of motions is a subgroup of the projective orthogonal group, since the latter contains the point and line reflections. This subgroup is the whole group in the elliptic case, and in the Euclidean case if Playfair's axiom is satisfied.

We can prove that the metric geometries are precisely the structures isomorphic to "metric subdomains" of projective metric geometries in the sense of KLOPSCH [9] and KINDER [6]. Thus, in the finite dimensional case, our approach agrees with Kinder's.

## IV. Lattice theoretic approach to elliptic geometry

Consider a nonempty set r and an antireflexive symmetric relation $\sim$ on r. Assume the following Axioms:

(G1) $\forall o,p,q\ [o \neq p \rightarrow \exists s,t\ [q,s\sim t\ \&\ \forall u\,[o,p\sim u \leftrightarrow s,t\sim u]]]$

(G2) $\exists o,p,q\ [o\sim p\sim q\sim o]$

(G3) $\forall o,p\ [o \neq p \rightarrow \exists s,t\ [q \neq o,p\ \&\ \forall u\,[o,p\sim u \rightarrow q\sim u]]]$

If $p\in r$ and $x\subseteq r$, define $p\sim x \leftrightarrow (\forall q\in x)\ p\sim q$, and $x^{\perp}=\{p:p\sim x\}$. Then $x\rightarrow x^{\perp\perp}$ is a closure operator on r; let $F^{\perp\perp}$ denote the lattice of subsets of r closed under this operator. KINDER & WOLFF [7] showed that $F^{\perp\perp}$ is atomic - its atoms are the singleton subsets of r. Moreover, the joins of finite sets of atoms form a modular sublattice of $F^{\perp\perp}$.

If $y\subseteq r$, denote by $\overline{y}$ the union of all subsets $x^{\perp\perp}$ for finite $x\subseteq y$. Then $y\rightarrow\overline{y}$ is also a closure operator; let $\overline{F}$ denote <u>its</u> lattice of closed subsets. If $x\subseteq r$ is finite, then $\overline{x}=x^{\perp\perp}$. Thus, in general, $\overline{y}$ is the union of all subsets $\overline{x}$ for finite $x\subseteq y$. We can prove that $\overline{F}$ is modular. From these considerations and Axioms G2 and G3, it follows that by defining "line" and



"orthogonality of lines" as usual, we can make r into an elliptic orthogonal geometry.

Obvious additional definitions and postulates can be stated to ensure that r is an elliptic metric geometry.

## V. Group theoretic approach to elliptic geometry

Consider a group g with a generating set r invariant under inner automorphisms and consisting of involutions. Use the notion $o\sim p$ to indicate that are distinct commuting elements of r. Assume Axioms G1 to G3 of § IV and

(G4) $\forall o,p,q\ [\forall u\,[o,p\sim u \rightarrow q\sim u] \rightarrow \exists s\quad opq = s]$.

Then r with the relation $\sim$ defines an elliptic orthogonal geometry. Moreover, $p\rightarrow opo$ is a reflection in each $o\in r$, hence by G4 the geometry is elliptic metric!

Following KINDER [8] we can prove that if $p_0,\dots,p_m\in r$ and $p_0\dots p_m$ belongs to the center of g, and m is the smallest integer for which there exist such p's, then $p_0\sim\dots\sim p_m$ and m is the dimension of the entire space. Thus, to ensure that the center of g is trivial, and that g is isomorphic to its inner automorphism group, which is the group of motions of the geometry, we merely need to assume the following infinite sequence of axioms ($G5_m$) for m = 3, 4,...:

$\forall p_0,\dots,p_m\quad [p_0\sim\dots\sim p_m \rightarrow \exists o\quad [op_0\dots p_m \neq p_0\dots p_m o]]$

In the presence of an axiom guaranteeing finite dimension m, only $G5_m$ needs to be postulated; in the presence of axioms guaranteeing infinite dimensionality, axioms $G5_m$ are redundant.

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