Suppose the discontinuities of u are at the points t_i. Then (y_n, u) is a linear combinations of terms

(15) $$\frac{1}{2\pi}\int_{t_i}^{t_{i+1}} y_n(s)ds = z_n(t_{i+1}) - z_n(t_i).$$

Hence $\lim(y_n, u) = 0$ and, by (14), $N_\varepsilon > N$ can be so chosen that $\| y_n \| < 2\varepsilon$ for $n > N_\varepsilon$. Therefore $\lim \| y_n \| = 0$ is proved.

From the C-completeness of the system $\{e_k\}$ we obtain a simple proof for a Fourier convergence theorem. Suppose f is a function of period 2π which has a piecewise continuous (more generally, a square-summable) derivative f'. Put

(16) $$r_{m,n} = f - \sum_{k=-m}^{n} (f, e_k)e_k \qquad m, n = 0, 1, \cdots.$$

Since $f(\pi - 0) = f(-\pi + 0)$, integration by parts gives $(f', e_k)e_k = (f, e_k)e_k'$, hence $r'_{m,n} = f' - \sum_{k=-m}^{n}(f', e_k)e_k$. C-completeness of the system $\{e_k\}$ implies

(17) $$\lim_{m,n\to\infty} \| r_{m,n} \| = 0, \quad \lim_{m,n\to\infty} \| r'_{m,n} \| = 0.$$

This, in connection with the trivial identity

(18) $$tr_{m,n}(t) = \int_0^t r_{m,n}(s)ds + \int_0^t sr'_{m,n}(s)ds$$

gives $\lim_{m,n} r_{m,n}(t) = 0$ for each $t \neq 0$, but also for $t = 0$ since $r_{m,n}(2\pi) = r_{m,n}(0)$. This proves pointwise convergence of the Fourier series to f. Moreover,

(19) $$| r_{m,n}(t) - r_{m,n}(-\pi) | = \left| \int_{-\pi}^{t} r'_{m,n}(u)du \right| \leqq \| r'_{m,n} \|,$$

and since $\lim r_{m,n}(-\pi) = 0$ and $\lim \| r'_{m,n} \| = 0$, (19) implies uniform convergence of the partial sums $\sum_{k=-m}^{n}(f, e_k)e_k$ to f.

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HAAR INTEGRALS ON TOPOLOGICAL RINGS

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Let R be a locally compact topological ring with identity. Denote by R^+ its additive group, and by $R^\times$ the multiplicative group of its units, and assume that $R^\times$ is open in R. We shall give a simple method of constructing a Haar integral on $R^\times$ from a given Haar integral on R^+. This result complements, and its proof is suggested by, the usual examples of Haar integrals. We then work out the particular example of the Haar integrals for the ring R of 2 by 2 real matrices.

Following Nachbin [2], we define a **right Haar integral** on a locally compact topological group G (notated additively) to be a nontrivial positive linear functional $\int$ on the vector space V of continuous real valued functions on G with compact support, such that for each $f \in V$ and $t \in G$,

$$\int f(x+t)\,dx = \int f(x)\,dx.$$

(Left Haar integrals may be defined and treated similarly.) Note that a right Haar integral on G always exists; further, if $\int_1$ and $\int_2$ are both right Haar integrals on G, then there exists a unique positive real number Δ such that $\int_1 = \Delta \int_2$. (We assume only these facts about the Haar integral, so it will be necessary to give a proof of a well-known property of the modulus function.)

Let R be a locally compact topological ring with identity, such that $R^\times$ is open in R. Then R^+ and $R^\times$ are locally compact topological groups under the topologies inherited from R. Let V^+ and $V^\times$ denote the vector spaces of continuous real valued functions with compact support on R^+ and $R^\times$, respectively. Let $\int^+$ be a Haar integral on R^+; we shall construct from $\int^+$ a right Haar integral $\int^\times$ on $R^\times$.

Let $t \in R^\times$. If $f \in V^+$, then the function that maps each $x \in R^+$ onto $f(xt)$ is also in V^+; define

$$\int_t^+ f(x)\,dx = \int^+ f(xt)\,dx. \tag{1}$$

THEOREM 1. *If $f \in V^+$, then the function that maps each $t \in R^\times$ onto $\int_t^+ f(x)\,dx$ is continuous.*

Proof. This results from the following inequality, which holds for all t and u in $R^\times$:

$$\left| \int_t^+ f(x)\,dx - \int_u^+ f(x)\,dx \right| \leqq \int^+ |f(xt) - f(xu)|\,dx.$$

THEOREM 2. *If $t \in R^\times$, then $\int_t^+$ is a Haar integral on R^+.*

Proof. Clearly, $\int_t^+$ is a nontrivial positive linear functional on V^+. Moreover, if $u \in R^+$, then

$$\int_t^+ f(x+u)\,dx = \int^+ f(xt+ut)\,dx = \int^+ f(xt)\,dx = \int_t^+ f(x)\,dx.$$

By Theorem 2, for each $t \in R^\times$ there exists a unique positive real number $\Delta(t)$ such that for each $f \in V^+$,

$$\int^+ f(x)\,dx = \Delta(t) \int_t^+ f(x)\,dx. \tag{2}$$

We call $\Delta(t)$ the **modulus of** t.

THEOREM 3. *The modulus function Δ is a continuous homomorphism from $R^\times$ to the multiplicative group of positive real numbers* [**2**, p. 77].

Proof. Continuity results from Theorem 1: use some $f \in V^+$ such that $\int^+ f(x)\,dx \neq 0$. For this f and any t and u in $R^\times$,

$$\begin{aligned}\Delta(tu)^{-1}\int^+ f(x)\,dx &= \int^+_{tu} f(x)\,dx = \int^+ f(xtu)\,dx = \int^+_u f(xt)\,dx \\ &= \Delta(u)^{-1}\int^+ f(xt)\,dx = \Delta(u)^{-1}\int^+_t f(x)\,dx \\ &= \Delta(u)^{-1}\Delta(t)^{-1}\int^+ f(x)\,dx.\end{aligned}$$

Thus $\Delta(tu) = \Delta(t)\Delta(u)$.

If $f \in V^\times$, then the support of f excludes a neighborhood of 0 in R^+, hence we can extend the function f/Δ to a function in V^+ by setting $f(x)/\Delta(x) = 0$ for each $x \in R^+ - R^\times$. Then we define

(3) $$\int^\times f(x)\,dx = \int^+ f(x)\Delta(x)^{-1}\,dx.$$

THEOREM 4. *$\int^\times$ is a right Haar integral on $R^\times$.*

Proof. Clearly, $\int^\times$ is a nontrivial positive linear functional on $V^\times$. Moreover, if $t \in R^\times$, then

$$\begin{aligned}\int^\times f(xt)\,dx &= \int^+ f(xt)\Delta(x)^{-1}\,dx = \Delta(1/t)\int^+_{1/t} f(xt)\Delta(x)^{-1}\,dx \\ &= \Delta(t)^{-1}\int^+ f(x)\Delta(xt^{-1})^{-1}\,dx = \Delta(t)^{-1}\int^+ f(x)\Delta(x)^{-1}\Delta(t)\,dx \\ &= \int^+ f(x)\Delta(x)^{-1}\,dx = \int^\times f(x)\,dx.\end{aligned}$$

Example: the ring R of 2 by 2 real matrices. Here $R^\times$ is the group of invertible 2 by 2 real matrices, and $\int^+$ is the Lebesgue integral on real 4-space. We determine first the modulus function:

(4) $$\Delta(t) = (\det t)^2.$$

This equation arises from a calculation with Jacobians: if $y = xt$ and the matrices x and y have entries x_{ij} and y_{ij}, respectively, then

(5) $$\int^+ f(y)\,dy = \int^+ f(xt)\,J\,dx = J\int^+ f(xt)\,dx,$$

(6) $$J = \frac{\partial(y_{11}, y_{12}, y_{21}, y_{22})}{\partial(x_{11}, x_{12}, x_{21}, x_{22})} = (\det t)^2.$$

Equation (4) then follows from (1), (2), (5), and (6). The Haar integral on $R^\times$ is given by Equations (3) and (4):

(7) $$\int^\times f(x)\,dx = \int^+ \frac{f(x)}{(\det x)^2}\,dx.$$

Note: This result generalizes theorems in Bourbaki [**1**, p. 33] and Weil [**3**, p. 89].

This work was supported by the National Research Council of Canada. The author acknowledges a suggestion of Professor Steven A. Gaal.

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GREGORY'S METHOD FOR NUMERICAL INTEGRATION

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Recently Peters and Maley [**1**] obtained formulas of the form

(1) $$h \sum_{i=0}^{n} f_i - h \sum_{j=0}^{m} A_j^m (f_j + f_{n-j}),$$

with $m \leqq n$, for approximating to the integral

$$\int_{x_0}^{x_n} f(x)\,dx.$$

In (1) the abscissas x_j are equally spaced, with $x_j = x_0 + jh$, $j = 0, 1, \cdots, n$, and f_j denotes $f(x_j)$. These integration rules are exact if $f \in \Pi_m$, the set of polynomials of degree not greater than m. In [**1**], for a given value of $m \leqq n$, each rule (1) is constructed by adding together contributions from the intervals $[x_0, x_j]$ and $[x_{n-j}, x_n]$ for $1 \leqq j \leqq m-1$ and $[x_j, x_{j+m}]$ for $0 \leqq j \leqq n-m$. Each contribution gives exact results for integrands $f \in \Pi_m$. This ingenious 'overlapping' method gives m times the required integral.

We shall show here that for m even, say $m = 2k$, the formulas (1) may be expressed in the form

(2) $$h \sum_{i=0}^{n} f_i + h \sum_{i=0}^{2k} a_i (\Delta^i f_0 + (-1)^i \nabla^i f_n),$$