Also solved by JOE MARCHIONE, Delaware County C.C.; ZUN SHAN and EDWARD T. H. WANG (jointly), Wilfrid Laurier U.; and the proposer.

$n^{\sqrt{n+1}}$ Versus $(n+1)^{\sqrt{n}}$

360. (Sept. 1987) *Proposed by J. H. Webb, University of Capetown, South Africa*

Prove that, if n is a natural number,

$$n^{\sqrt{n+1}} < (n+1)^{\sqrt{n}} \quad \text{for } n < 7 \tag{i}$$

$$n^{\sqrt{n+1}} > (n+1)^{\sqrt{n}} \quad \text{for } n \geq 7. \tag{ii}$$

Solution I by Eugene Levine, Adelphi University, Garden City, NY

More generally, let a, b, and t denote positive reais with $a < b$. The function $\ln x/x$ is strictly increasing for $0 < x \leq e$, so if $b \leq e^{1/t}$, then $\ln a^t/a^t < \ln b^t/b^t$, or $b^t \ln a < a^t \ln b$. Thus if $0 < a < b \leq e^{1/t}$, then $a^{b^t} < b^{a^t}$. Furthermore, for $x \geq e$ the function $\ln x/x$ is strictly decreasing, so for $a \geq e^{1/t}$ the above inequalities are reversed: Thus if $e^{1/t} \leq a < b$, then $a^{b^t} > b^{a^t}$.

For the given problem, substitution of the values $a = n$, $b = n+1$, and $t = 1/2$ (so that $7 < e^{1/t} < 8$) in the above inequalities will give all the required results except when $n = 7$. In that case, numerical calculation to the nearest tenth gives $7^{\sqrt{8}} = 245.6$ and $8^{\sqrt{7}} = 245.1$ and hence, for all natural numbers n, the required results are proven.

Solution II by James T. Smith, San Francisco State University, CA

For $n \geq 8$, apply the natural logarithm twice on each side of (ii) and rearrange the resulting inequality to find

$$\frac{\ln\ln(n+1) - \ln\ln(n)}{\ln(n+1) - \ln(n)} < 1/2.$$

By the Cauchy mean value theorem, there exists $n < \xi < n+1$ such that the quotient on the left equals

$$\left.\frac{[d/dx]\ln\ln x}{[d/dx]\ln x}\right|_{x=\xi} = \frac{1}{\ln\xi} < \frac{1}{\ln 8} < 1/2,$$

which proves the result. For $n = 1, 2, \ldots, 7$, result (i) is easily verified using a calculator.

Also solved by CHARLES ASHBACHER, Mount Mercy C.; BRUCE S. BABCOCK, Pennsylvania State U.; SEUNG-JIN BANG, Seoul, Korea; FRANK P. BATTLES, Massachusetts Maritime Academy; ROBERT E. BERNSTEIN, Manitowoc, WI; MARK BIEGERT, Honeywell, Inc.; DAVID BOYD, Valdosta State C.; JAMES E. CARPENTER, Iona C.; CHICO PROBLEM GROUP, California State U.; GEORGE DAY, Allegheny C.; RICHARD DELAWARE, U. of Missouri-Kansas City; GORDON FISHER, James Madison U.; RICHARD A. GIBBS, Fort Lewis C.; PAUL M. HARMS, Taylor U.; J. HEUVER, Grande Prairie Composite H.S.; GEOFFREY A. KANDALL, Hamden, CT; BENJAMIN G. KLEIN, Davidson C.; KEN KORBIN, NYC Technical C.; ROBERT LAVELLE, Iona C.; DENNIS M. LUCIANO (two solutions), Western New England C.; DAVID E. MANES, SUNY at Oneonta; JOE MARCHIONE, Delaware County C.C.; JACK McCOWN, Central Oregon C.C.; BOB PRIELIPP, U. of Wisconsin-Oshkosh; NORMAN SCHAUMBERGER, Bronx C.C.; H.-J. SEIFFERT, Berlin, Germany; M. SELBY, U. of Windsor; ZUN SHAN and EDWARD T. H. WANG (jointly), Wilfrid Laurier U.; TIMOTHY SIPKA, Anderson U.; JAN SODERKVIST, Sweden; ROBERT S. STACY, NY; EDWARD F. VITEK, Schenectady, NY; MICHAEL VOWE, Therwil, Switzerland; JOSEPH WEINER, Pan American U.; HARRY WEINGARTEN, Sunnyvale, CA; and the proposer.

Editor's Note: Schaumberger remarks that substitution of the values $a = e^2$, $b = \pi^2$, and $t = 1/2$ gives $(e^2)^\pi > (\pi^2)^e$, and hence another proof that $e^\pi > \pi^e$. Day remarks that consideration of the graph of $\ln x/\sqrt{x}$ and a few calculations show that $(3, 27)$ and $(4, 16)$ are the only positive integer solutions (u, v)