Example 4. The derivative of the integral

$$\int_0^\infty \frac{dx}{x^2+p} = \frac{\pi}{2} p^{-1/2}, \qquad p > 0 \tag{6}$$

with respect to p contains $\left(x^2+p\right)^2$ in the denominator and is therefore a uniformly convergent integral for $p \geq \varepsilon$, where ε is any positive number. Hence, we may successively differentiate (6) according to (1) and obtain

$$\int_0^\infty \frac{dx}{\left(x^2+p\right)^{n+1}} = \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^n n!}\cdot\frac{\pi}{2} p^{-(2n+1)/2}.$$

Multiplying the numerator and denominator by $2\cdot 4\cdot 6\cdot\cdots\cdot 2n$ and putting $p = 1$ gives

$$\int_0^\infty \frac{dx}{\left(x^2+1\right)^{n+1}} = \frac{(2n)!}{4^n\,(n!)^2}\cdot\left(\frac{\pi}{2}\right).$$

The substitution $x = \tan\theta$ changes this result to Wallis's integral

$$\int_0^{\pi/2} \cos^{2n}\theta d\theta = \frac{(2n)!}{4^n\,(n!)^2}\cdot\left(\frac{\pi}{2}\right).$$

References

1. R. Courant, *Introduction to Calculus and Analysis*, Vol. 1, Springer-Verlag, 1989.
2. J. Wiener, An analytical approach to a trigonometric integral, *Missouri Journ. of Math. Sci.*, 2 (2), 1990, 75–77.

Geometry at Work

James T. Smith (San Francisco State University, smith@math.sfsu.edu) sends an example of a former application of mathematics, from James Fenimore Cooper's *The Prairie: A Tale* (chapter 9):

> The pursuit of a bee-hunter is not uncommon, on the skirts of American society. … When the bees are seen sucking the flowers, their pursuer contrives to capture one or two. He then chooses a proper spot, and suffering one to escape, the insect invariably takes its flight towards the hive. Changing his ground to a greater or less distance, according to the circumstances, the bee-hunter then permits another to escape. Having watched the courses of the bees, which is technically called lining, he is enabled to calculate the intersecting angle of the two lines, which is the hive.