Definitions and Nondefinability in Geometry: Pieri and the Tarski School

James T. Smith
San Francisco State University

Abstract[1]

In 1886 Mario Pieri became professor of projective and descriptive geometry at the Royal Military Academy in Turin and, in 1888, assistant at the University of Turin. By the early 1890s he and Giuseppe Peano, colleagues at both, were researching related questions about the foundations of geometry. In 1889 Pieri published an annotated translation of G. K. C. von Staudt's fundamental 1847 work on projective geometry. Peano's 1889 booklet on foundations of geometry divorced that discipline from the study of the real world:

> Depending on the significance attributed to the undefined symbols ... the axioms can be satisfied or not. If a certain group of axioms is verified, then all the propositions that are deduced from them will be equally true ... (page 24).

This freedom, and the distinction between syntactic properties of symbols and their semantic relationships to the objects they denote, was essential for all later studies of definability. Both colleagues became intrigued with Staudt's arguments. In 1890 Peano repaired a lapse in Staudt 1847. Pieri soon became a preeminent member of the Peano School. During the next two decades, he used, refined, and publicized Peano's logical methods in several major studies of foundations of geometry. A series of his papers culminated in Pieri 1898, the first complete axiomatization of real projective geometry.

In 1894 Peano introduced *direct motion* as an undefined notion in Euclidean geometry. A geometric transformation, it does not involve time. Figures can be defined as *congruent* just when some direct motion maps one to the other. Pieri pursued this idea deeply in his *Point and Motion* memoir, 1900a, and adapted its approach for his 1908 *Point and Sphere* memoir. The present paper focuses on the history of those two studies and their influence as the root of later work of Alfred Tarski and his followers. It emphasizes Pieri's achievements in expressing Euclidean geometry with a minimal family of undefined notions, and in requiring set-theoretic constructs only in his treatment of continuity. It is adapted from and expands on material in the 2007 study of Pieri by Elena Anne Marchisotto and the present author.

In *Point and Motion* Pieri developed absolute geometry, a common part of Euclidean and hyperbolic geometry independent of continuity considerations, using only two undefined notions, *point* and *direct motion*. Some of his definitions were straightforward.

---

[1] For the congress on *The School of Giuseppe Peano between Mathematics, Logic, and Interlingua* to be held 6–7 October 2008 in Torino

For example, he called a point set *collinear* just when its members are all fixed by some single nontrivial direct motion. His definition of *betweenness* was more complicated and quite ingenious. In the related paper 1900b, Pieri suggested that this work could be adapted to provide a full axiomatization of Euclidean geometry based solely on *point* and the undefined relation, *equidistance* of two points from a third.

After a long struggle, Pieri finally obtained appointment as university professor in 1900 at Catania. There he completed that project with his *Point and Sphere* memoir. Its most notable definitions derive from equidistance the notions of *collinearity* (as already suggested by Leibniz in [1679] 1971) and of *congruence* of point pairs. From those he defined *isometry* and *direct motion*, then proceeded as in *Point and Motion*. The present paper includes details of Pieri's definitions. His axioms were frightfully complicated, but would now be called first-order, except for the continuity axioms. Moreover, Pieri published all details of his proofs!

Although *Point and Sphere* had received little explicit attention, during the 1920s Tarski noticed its minimal set of undefined notions, its extreme logical precision, and its use of only a restricted variety of logical methods. Those features permitted him to adapt and reformulate Pieri's system in the context of first-order logic, which was then emerging as a coherent framework for logical studies. Tarski's theory was much simpler, and encouraged deeper investigations of the metamathematics of geometry.

In particular, Tarski and Adolf Lindenbaum pursued the study of definability, extending the 1900 work of Alessandro Padoa, a member of the Peano School. In two 1935 papers, they used this technique to show that *point* and *betweenness* cannot serve as the sole undefined notions in Euclidean geometry, counter to the 1904 claim of Oswald Veblen, and that Pieri's selection of point and equidistance was optimal. No family of binary relations, however large, can serve as the sole undefined relations. Details of these arguments are included in the present paper.

No full description of Tarski's system was published for decades, but his 1959 summary stimulated research activity. Tarski's followers have extended his methods to apply to other geometric theories as well as the Euclidean. The present paper concludes with a description of the 1990–1991 discovery by Victor Pambuccian that Euclidean geometry can be based on a single binary relation if the underlying logic is strengthened to admit countably infinite conjunctions. The geometric aspects of Pambuccian's argument would have been familiar to Pieri in 1908!

Bibliography

Henkin, Leon, Patrick Suppes, and Alfred Tarski, editors. 1959. *The Axiomatic Method with Special Reference to Geometry and Physics: Proceedings of an International Symposium Held at the University of California, Berkeley, December 26, 1957–January 4, 1958*. Amsterdam: North-Holland Publishing Company.

International Congress of Philosophy. 1900–1903. *Bibliothèque du Congrès International de Philosophie*. Four volumes. Paris: Librairie Armand Colin.

Leibniz, Gottfried Wilhelm von. [1679] 1971. Characteristica geometrica. In Leibniz [1849–1863] 1971, part 2, volume 1, 141–211.

———. [1849–1863] 1971. *Leibnizens mathematische Schriften*. Two parts, with seven volumes in all. Olms paperbacks, 41–47. Edited by K. I. Gerhardt. Hildesheim: Georg Olms Verlag, 1971. Originally published during 1849–1863 in Berlin by A. Asher.

Lindenbaum, Adolf, and Alfred Tarski. 1935. Über die Beschränktheit der Ausdrucksmittel deduktiver Theorien. *Ergebnisse eines mathematischen Kolloquiums* 7: 15–22.

Marchisotto, Elena Anne, and James T. Smith. 2007. *The Legacy of Mario Pieri in Geometry and Arithmetic*. Boston: Birkhäuser.

Padoa, Alessandro. [1900] 1901. Essai d'une théorie algébrique des nombres entiers, précédé d'une introduction logique à une théorie déductive quelconque. In International Congress of Philosophy 1900–1903, volume 3 (1901), 309–365.

Pambuccian, Victor. 1990–1991. Unit distance as single binary predicate for plane Euclidean geometry. *Zeszyty Naukowe—Geometria* 18: 5–8, 19: 87.

Peano, Giuseppe. 1889. *I principii di geometria logicamente esposti*. Torino: Fratelli Bocca.

———. 1890. Sopra alcune curve singolari. *Atti della Reale Accademia delle Scienze di Torino* 26(1890–1891): 299–302.

———. 1894. Sui fondamenti della Geometria. *Rivista di matematica* 4: 51–90.

Pieri, Mario, editor and translator. 1889. *Geometria di posizione*, by G. K. C. von Staudt. Preceded by a study of the life and works of Staudt by Corrado Segre. Biblioteca matematica, 4. Turin: Fratelli Bocca Editori. Edited translation of Staudt 1847.

———. 1898. I principii della geometria di posizione composti in sistema logico deduttivo. *Memorie della Reale Accademia delle Scienze di Torino* (series 2) 48: 1–62.

———. 1900a. Della geometria elementare come sistema ipotetico deduttivo: Monografia del punto e del moto. *Memorie della Reale Accademia delle Scienze di Torino* (series 2) 49: 173–222.

———. 1900b. Sur la géométrie envisagée comme un système purement logique. In International Congress of Philosophy 1900–1903, volume 3 (1901), 367–404.

———. 1908. La geometria elementare istituita sulle nozioni di "punto" e "sfera". *Memorie di matematica e di fisica della Società Italiana delle Scienze* (series 3) 15: 345–450. Translated in Marchisotto and Smith 2007, chapter 3.

Staudt, Georg Karl Christian von. 1847. *Geometrie der Lage*. Nuremberg: Verlag der Fr. Korn'schen Buchhandlung. Pieri 1889 is an annotated translation of this work.

Tarski, Alfred. 1935. Einige methodologischen Untersuchungen über die Definierbarkeit der Begriffe. *Erkenntnis* 5: 80–100.

———. [1957] 1959. What is elementary geometry? In Henkin, Suppes, and Tarski [1957] 1959, 16–29.

Veblen, Oswald. 1904. A system of axioms for geometry. *Transactions of the American Mathematical Society* 5: 343–384.