Outline20080317.oo3

Topic

- Introduction
- Basics of Banach Spaces
  - Assume the reader is familiar with the definition of vector space
  - **Definition** of Norm
    - A function n from X to R with the following properties
      - $n(x) \geq 0$
      - $n(x)=0$ iff $x=0$
      - $n(x+y)\leq n(x)+n(y)$
      - $n(ax)=|a|n(x)$
  - **Definition** of complete
    - A complete space is a space where every Cauchy sequence converges to an element in the space (in norm).
    - Under the metric induced by the norm $d(x,y)=n(x-y)$
  - **Definition** of Banach Space
    - A complete normed linear space.
    - Usually the ground field is over R, but we can consider it over C. However, nothing is really added and this can be broken down to the case of just over R (Singer)
    - X and Y will only refer to Banach Spaces from now on
  - **Definition** Bounded operator
    - An linear function T from $X \to Y$ is bounded if $n(Tx) \leq M n(x)$ for all x in X. Where the norms n are in the respective spaces
    - The smallest constant M which this is true for is call the norm of an function
  - **THM:** TFAE (Carothers p11)
    - T is continous at 0
    - T is continuous
    - T is uniformly continuous
    - T is Lipschitz
    - T is bounded
  - ?Closed Graph **Thm**
  - Open Mapping thm
    - **Thm**: If T is a continuous one-to-one linear operator from X onto Y then \inv{T} is also continuous. (Royden p230)
  - Hahn-Banach thm
    - **Thm:** If f is a linear functional on a linear subset L of a Banach space X, then there is a linear function F on all of X s.t. $\norm{f}=\norm{F}$ and $f(x)=F(x)$ for all $x \in L$ (James p626)
- Schauder vs. Hamel Basis
  - **Definition** of Hamel Basis
    - A set S in which every element can be uniquely written by a *finite* linear combination of elements from S
  - **Theorem:** Every Vector Space Has a Hamel basis
    - **Proof:** Notice that Independent sets make a Poset
      - By Zorn's Lemma, there is a Maximal Element S
      - This S is the basis
  - **Definition** of Schauder Basis
    - A sequence $\{x_n\}$ in which every element can be written uniquely by a linear combinations of elements from the sequence, not necessarily finite
    - This convergence is in norm (not just pointwise)
  - The two concepts are the same in finite dimensional spaces but major differences can come about in infinite dimensional cases.
  - Why we need to use Schauder Basis
    - Hamel basis tells us nothing of how to write an element.
    - In the proof, since we used Zorn's lemma, the basis by its very nature is nonconstructive.
    - Hamel Basis are uncountable (by the Baire category thm) (carothers p25) and hence unwieldy
    - Basis from now on will refer to Schauder Basis

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- Basis Problem
  - Does every Banach Space have a Schauder Basis?
    - Asked by Banach (Caraothers p 25)
  - First need to be a separable space
    - This must happens since by by definition of Schauder Basis, the x_n must converge in norm, and hence the span of x_n are dense in X. (Carothers p 24)
    - \ell_\infty does not have a basis since since not separable
      - ?Proof
  - Examples of Schauder Basis
    - The usual basis is a Schauder Basis for ℓ_p and c_0 (Carothers p 24)
      - Show this after we have some machinery later
    - C[0,1]
      - Harder than just the Polynomials
      - To see this take a function without derivative at 0
        - Cannot write it as a sum of polynomials
      - **Proof** for actual basis (James p627)
        - Order the dyadic numbers in the following way
        - {0,1,1/2,1/4,3/4,1/8,3/8,5/8,7/8,1/16...}
        - Let t_n be the nth number in the list
        - Let f_0 =1, f_1=t, f_n(x)=1 if x=t_n f_n be linear increasing for t_n-1<x<t_n and linear decreasing for t_n<x<t_n+1 and 0 otherwise
        - Next since the dyadic numbers are dense in R then for any function f, we can approximate it by a polygonal function
        - Insert a picture to make this easier to see
    - L2[0,1] (James p628)
      - Haar System
      - **Proof:** $h_0=1 h_n(x)=1 if t_n-1<x<t_n -1 if t_n<x<t_n+1$ and 0 otherwise
      - ...
      - Note that integrating these gives a basis for C[0,10
  - Hard to find, but still possible?
    - No, solved by Enflo (Enflo 1973)
    - Proof is beyond the scope of this paper
- Sufficient Conditions for a basis to exist
  - **Definition**: If $\{x_n\}$ is a basis for X then $P_n(x)=\sum_{i=1}^n a_ix_i$. i.e P_n is the projection of x onto the \span(x_1, \ldots x_n}$. (Carothers p26)
  - **Lemma**: If \{x_n\} is a basis for a Banach Space X then every P_n is continuous
    - **Proof**: Define a new norm $N(x)=\sup_n n(P_n(x))$ over all n
    - We want to show that Pn are bounded linear functions.
    - Clearly the identity operator on X with the N norm to X with the usual norm is continuous since it is bounded.
    - By the open mapping thm, since the identity operator is bijective need only show that it is countinous, but this is iff X is a Banach Space under the new Norm N.
    - Let y_k be Caucy in n norm
    - But $n(P_n(y_i)-P_n(yj)) \leq N(y_i-y_j) for all n
    - Let z_n=lim_{k \to \infty} P_ny_k
    - So $n(z_n-z_m)\leq n(z_n-P_ny_k)+n(P_ny_k-P_my_k)+n(P_my_k-z_m)
    - Each one of the terms of the inequality can be made arbitrarily small by the above
    - Let z=lim z_n in n norm.
    - Since $P_nz=z_n$ then P_n is countionus on a finite space
    - Thus $P_n(z_m)=z_{\min\{m,n\}}$.
    - Thus $z=\sum a_ix_i$.
  - **Thm**: A sequence \{x_n\} of nonzero vectors is a basis for the Banach Space X iff (carothers p27)
    - (i) The span of $\{x_n\}$ is dense in X
    - (ii) n(\sum_{i=1}^n a_ix_i) \leq M n(\sum_{i=1}^m a_ix_i) for all $n<m$ and some constant M
    - M is called the basis constant

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    - **Proof**: If n<m then n(\sum_{i=1}^n a_ix_i) =n(P_n(\sum_{i=1}^m a_ix_i)) \leq n(P) n(\sum_{i=1}^ma_ix_i)
    - Suppose (i) and (ii) hold. Let S be the $\span\{x_n\}
    - Inductively all the x_i are linearly independent
    - ...
  - Show that the usual basis is a basis for \ell_p and c_0
- Approximation property
  - **Definition** of a Finite Rank operator
    - A linear map T s.t T(X) is finite dimensional.
  - **Definition** of Approx Property: For every compact set K \subset X \epsilon>0 there is a operator T of finite rank s.t. \norm{Tx-x} \leq \epsilon for every x \in K.
  - If a Banach Space has a Schauder basis, it has the approximation property
    - Proof (James p638)
  - G gave many conjectures equivalent to every Banach Space has the approximation property (James p 638) (Pietsch 5.7.4)
    - Some of the conjectures
    - ...
  - Enflo's proof, actually showed that there is a space without the Bounded approximation property *(citation needed)*
  - In fact, after Enflo, many more spaces were found that did not have a basis. Surprisingly, some of them are subspaces of ℓ_p! (citation needed)
  - ?Bounded approximation property
- Basic Sequences
  - Can we hope to show that some subspace of a Banach Space X contains a Basis?
  - **Definition**: A sequence \{x_n\} is a basic sequence if $\{x_n\}$ is a basis for it's closed linear span.
  - Does every space have a basic sequence?
    - YES!
    - Proverd by Mazur (which has also known to Banach) (Carothers p34)
  - **Lemma**: Let F be a finite dimensional subspace of a infinite dimensional space X. Then given $\epsilon>0$ there is an $x \in X$ with $\norm{x}=1$ s.t. $\norm{y}\leq (1+\epsilon) \norm{y-\lambda xl$ for all λ and $y \in F$.
    - **Proof**: Since F is finite dimensional then the unit ball is compact in X. Thus we can choose finitely many $y_k $ for the unit ball which are covered by the $\epsilon/2$ balls.
    - Chooses a norm one functional y_i* (by Hahn-Banach) s.t. $y_i*(y_i)=1$
    - Since $\cap ker y_i* is a subspace of a finite dimensional space, the codimension is also finite.
    - Thus there is a nonzero x of norm one that will make this ture
    - Now we get
    - ...
  - **Thm**: Every infinite dimensional Banach Space contains a closed subspace with a basis (carothers p35)
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- Conclusion
- ?Things to Add?
  - ?Every Banach Space Embeds Isometrically into a space with a basis
  - ?Dual Spaces
  - ? Properties of c_0, \ell_2 \ell_1
  - ?Unconditional Bases
  - ? James Space
  - ? Enflo's Solutions and other solutions since then