This *WordPerfect* document is the current state of an outline, composed in class, of a model expository paper. During each discussion I'll edit it as appropriate, in a new temporary copy. After the discussion, I'll update this document from that, and include a "snapshot" of its current state in the corresponding class outline.

Where to place historical information is undecided. In this document there are tentative placeholders where it might be appropriate to include that. But this strategy may result in too many digressions, particularly if no coherent story line should be discovered to tie the history together. Should that be the case, these items should go instead in the history suggested near the end of the paper.

1. This line is merely a placeholder to make the indentation correspond to that in the snapshots.

Dimension in Linear Algebra and Related Theories

a. *Introduction*
    i.   This paper presents the main features of the theory of linear dependence.
         (1) This theory includes several of the major concepts and theorems of linear algebra,
         (2) commonly introduced in first courses in linear algeba.
    ii. But its context is considerably more general than that,
         (1) to permit a much broader scope of application.
    iii. The theory is presented in the context of a vector space V over a division ring K of scalars.
         (1) Footnote: an alternative term for *division ring* is *skew field*.
         (2) Postulates for a division ring.
         (3) Examples: real numbers, complex numbers, rational numbers. (Others later.)
         (4) Postulates for a vector space V over a division ring K, whose elements are called *scalars*.
         (5) Examples: real and complex n-space. (Others later.)
         (6) Definition of a subspace of V.
         (7) Examples: $\{0\}$, V, lines and planes given by parametric equations in any n-space, hyperplanes and hyperlines given by linear equations in any n-space.
    iv. The presentation follows that of Van der Waerden [1937] 1953.
         (1) In this framework,
              (a) dependence of a vector on a set of vectors is defined first,
              (b) and few basic theorems (¿ ∃ A BETTER TERM ?) are proved.

          (c)   All subsequent definitions and theorems are based solely on these.

      (2)   This permits all these definitions and theorems to be carried intact into the development of any theory that includes an analogous concept of dependence that satisfies those few basic theorems.

      (3)   Moreover, by eliminating distracting use of methods specific to vector spaces, the framework encourages presentation of the simplest proofs.

   v.   The present context for this theory is in fact considerably more general than that of a first linear-algebra course.

      (1)   Rather than requiring scalars to be real or complex numbers,

      (2)   it permits use of scalars in some finite division rings, such as $\mathbb{Z}_p$ for prime p. The two-element ring $\mathbb{Z}_2 = \{0, 1\}$ is often used in the theory of computation.

      (3)   And it even permits the scalars to lie in a noncommutative division ring, such as the quaternions. This permits use of linear algebra at an earlier stage in the study of the foundations of geometry than would be possible otherwise, since commutativity of scalars is hard to derive from geometric axiom systems.

      (4)   Rather than requiring the vector space to be finite-dimensional, the proofs presented here apply even to infinite-dimensional spaces. This permits use of linear algebra in studying the spaces of polynomials, formal power series, and continuous functions, for example. The examples could also include spaces of sequences of any finite length, of all sequences, of all convergent sequences, etc.

   vi.   MAYBE AN EXPLANATION IS NEEDED HERE THAT THIS PAPER WILL WILL NOT ATTEMPT TO COVER THE THEORY OF MODULES OVER RINGS, WHICH SATISFY ALL THE POSTULATES FOR VECTORS AND SCALARS EXCEPT ... .

   vii.   A PARAGRAPH IS NEEDED HERE TO ENUMERATE THE SUBSEQUENT SECTIONS OF THE PAPER.

 b. *Some history*

   i.   The notion of a vector space was first codified by Giuseppe Peano in [1888] 2000, section 72. That work was an introduction to vector calculus, under intense development at that time to support a myriad of applications in physics and engineering. Peano based his presentation on the pioneering but unwieldy [1844] 1994 system of Hermann Grassmann, which never saw much direct use.

   ii.   MAYBE A SPECIFIC EXAMPLE FROM PEANO [1888] 2000 COULD BE MENTIONED HERE.

   iii.   The strategy of identifying certain algebraic structures as worthy of intensive study—such as groups, rings, fields, and vector spaces—and gathering their properties into algebraic theories emerged during the 1910–1930 decades through the publications and lectures of Ernst Steinitz, Emmy Noether, Emil Artin, and others. They were attempting to make comprehensible a vast array of earlier algebraic studies, and provide a framework for their extension. This organization was popularized in the 1931 first edition of Van der Waerden [1937] 1953.

iv.   That book remained for decades the standard introduction to abstract algebra. But linear algebra, the theory of vector spaces, took a slightly different turn. It took form later, but had reached its present form in Birkhoff and Mac Lane 1941, the standard English text in that subject at least through the 1960s.

v.   The approach in Van der Waerden [1937] 1953, section 33, on which the present paper is based, was developed for presenting the theory of transcendental extensions of fields: for example, the field obtained from the rational field by adjoining all algebraic expressions involving rational numbers and π.

vi.   HISTORICAL STUDIES ABOUT THOSE BOOKS ARE AVAILABLE. IT MIGHT BE POSSIBLE TO USE THEM AND REPORT MORE ABOUT THEIR AUTHORS' MOTIVATIONS.

c.   *Van der Waerden's framework*

i.   Explain about the framework, in Van der Waerden [1937] 1953, section 33.

ii.   Define *dependence* of a vector v on a set S of vectors.

iii.   Example from $\mathbb{R}^3$.

iv.   Note that each $v \in S$ depends on S.

v.   Show that v depends on S if and only if it depends on a finite subset of S.

vi.   Show that if each member of a set R of vectors depends on S, then each vector that depends on R also depends on S.

vii.   *Exchange Lemma.* If a vector v is not dependent on a set S of vectors, but is dependent on $S \cup \{u\}$ for some vector u, then u is dependent on $S \cup \{v\}$.

    (1)   *Proof.* v would be the sum of a linear combination of vectors in X and tu for some scalar t. But t must be nonzero, so has an inverse, and u would thus be a linear combination of vectors in $S \cup \{v\}$.

    (2)   Example from $\mathbb{R}^3$.

viii.   SHOULD WE REPEAT HERE THAT WE'LL BASE ALL FURTHER RESULTS ONLY ON THE NOTION OF DEPENDENCE AND THE PROPERTIES JUST PROVED?

d.   *Bases*

i.   *Span and independence*

    (1)   Define *span* $\bar{S}$ of a set S of vectors.

    (2)   State and prove some properties of the mapping $S \rightarrow \bar{S}$. For example, that it's a closure operator, and that spans are subspaces.

    (3)   Define a set S of vectors to be *independent* if no $v \in S$ depends on $S - \{v\}$.

    (4)   Example from $\mathbb{R}^3$ that's not a basis.

ii.   *Theorem.* The following properties of a set B of vectors are equivalent:

    (1) S is independent and span the whole vector space V.

    (2) B is a maximal independent subset of V.

    (3) B is a minimal spanning subset of V.

     iii.   *Definition*. A subset with these properties is called a *basis* of V.

     iv.   Examples from spaces of various dimensions, including that of finite sequences of any length.

     v.   *Theorem*. Every independent set S of vectors is contained in a basis.

       (1)   *Proof*. Show that the family $\mathcal{F}$ of all independent sets containing S satisfies the hypotheses of Zorn's lemma. Zorn's lemma then implies that $\mathcal{F}$ has a maximal member.

     vi.   *Corollary*. Every vector space has a basis.

       (1)   *Proof*. The empty family is independent.

     vii.   Note that it's not clear what a basis of the space of *all* sequences might be.

     viii.   *Theorem*. Every set S of vectors contains a basis of the span of S.

       (1)   *Proof*. This is a little more complicated, and uses the exchange lemma.

     ix.   *Exchange Theorem*. If S, S' are bases then for every $b \in B$ there exists $b' \in B'$ such that $(B - \{b\}) \cup \{b'\}$ is a basis.

       (1)   *Proof*. This, too, uses the exchange lemma.

     x.   Examples from $\mathbb{R}^3$ and from some higher dimension.

     xi.   *Corollary*. All bases are equinumerous.

       (1)   The proof in my notes uses the axiom of choice to well-order the bases, then uses recursion on the order relations, with the exchange theorem, to construct a bijection from one base to the other.

       (2)   I SUSPECT THAT THIS CAN BE SHORTCUT TO USE ZORN'S LEMMA, ANOTHER CONSE-QUENCE OF THE AXIOM OF CHOICE, INSTEAD OF WELL-ORDERING AND RECURSION. THAT WOULD BE A GOOD TOPIC FOR THE WRITER TO INVESTIGATE.

     xii.   *History*.

       (1)   The central role of the exchange lemma was recognized by Ernst Steinitz in 1910, who was using these techniques to study transcendental extensions of fields. It was emphasized as a fundamental principle of linear algebra in Van der Waerden [1937] 1953, section 33.

       (2)   IT MIGHT BE INTERESTING TO REPORT WHETHER THE PRINCIPLE IS STILL REGARDED AS FUNDAMENTAL IN CONTEMPORARY TEXTS.

       (3)   The theorem that *every* vector space has a basis (regardless of its dimension) is due essentially to Georg Hamel in 1905. IT MIGHT BE INTERESTING TO DESCRIBE THE APPLICATION HAMEL WAS STUDYING.

       (4)   THE USE OF DIFFERENT BASES FOR THE SAME SPACE PROBABLY AROSE SIMULTANEOUSLY IN BOTH ANALYTIC GEOMETRY AND (ESPECIALLY IN HIGHER DIMENSIONS) IN THE STUDY OF SOLUTIONS SYSTEMS OF DIFFERENTIAL OR LINEAR EQUATIONS. WHILE THAT WOULD BE INTERESTING TO INVESTIGATE AND REPORT, THAT MIGHT BE TOO BIG A TASK FOR THIS PAPER.

   e.   *Dimension*

     i.   *Definition*. The dimension of a vector space is the common cardinality of its bases.

     ii.   Examples: K^n has dimension n. The space of finite sequences of any length has dimension ω. We might give the dimension 2^ω of the space

of all sequences of real numbers, but I don't remember now the proof of that. This would be a good question for the writer to investigate.

iii. *Corollary*. $\dim(X \cap Y) + \dim(X \vee Y) \leq \dim X + \dim Y$ for any subspaces X, Y.

(1) *Proof*. This comes from repeated use of the theorem about extending independent subsets to bases, then counting.

(2) My notes are set in a context more general than vector spaces. For vector spaces, equality may hold here. I'm not sure.

(3) Examples?

(4) My notes give some conditions under which equality does hold. I think they are met in the finite-dimensional case. Investigating the infinite dimensional case would be a good project for the writer.

iv. *Rank & nullity theorem*

(1) My notes provide a general proof that when X is a subspace of a space Y, then $\dim X + \dim(Y/X) = \dim Y$, where Y/X is, at least, closely related to the usual quotient space.

(2) I think that leads in the finite dimensional case to the theorem, if f is a linear map from one space V to another, then $\dim V = \dim$ Range $f + \dim$ Kernel of f.

(3) Examples?

(4) I DON'T KNOW WHETHER PRECISELY THAT THEOREM HOLDS IN INFINITE DIMENSIONAL LINEAR ALGEBRA. INVESTIGATING THAT WOULD BE A GOOD PROJECT FOR THE WRITER.

v. *History*. I DON'T KNOW ANYTHING ABOUT THE HISTORY OF THE DIMENSIONALITY RESULTS. IT MIGHT BE POSSIBLE TO FIND SOMETHING IN STANDARD HISTORICAL TEXTS.

f. *History*. SHOULD IT PROVE UNFEASIBLE TO INSERT THE HISTORY SECTIONS AS SUGGESTED ABOVE, THEY COULD GO HERE IN A SEPARATE HISTORICAL SECTION.

g. *Further applications*

i. Connections with matrix theory and solution of equations.

(1) This is mostly limited to finite dimensions.

(2) But there is a theory of infinite-dimensional matrices, where the vectors must be limited to infinite sequences of scalars that satisfy some summability criterion. Analysis results enter quickly here.

(3) If this matrix theory were to include determinants, the scalars must be commutative and $1+1 \neq 0$.

(4) Van der Waerden's framework can be applied in geometry directly: under certain conditions, a point is said to be dependent on a set of points. Various familiar properties of "flat" subspaces and their dimensions are deduced from geometric axioms.

(5) Van der Waerden in fact applied his framework to the study of the transcendence degree of a field extension.

h. *Conclusion*

i. The conclusion should consist of several sentences that rephrase the first points in the introduction,

ii. list what was done in the body (without being too dry),

  iii. and point out that the *Further Applications* section points the way to applying these results in several further areas of mathematics.

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