Title

1. Introduction
   a. Brief overview of the paper.
   b. History.

2. Preliminaries
   a. Axiom systems and ZF set theory. Independence.
   b. Choice functions in different strengths.
   c. When is a theorem equivalent to choice? How do we prove it?

3. Linear algebra theorems equivalent to choice (Bleicher)
   a. Introductory material
      i. 5 propositions for infinite dimensional vector spaces which are equivalent to some form of choice.
         1. *(Hamel Basis theorem) Every vector space V has a basis.*
         2. *If a subset of a vector space generates the space, then the subspace contains a basis.*
         3. *For each subspace S of V there is a disjoint subspace S' such that $S \cup S'$ generates V.*
         4. *Given a basis B of V and subspace S, there is a subset b' of B disjoint from S such that $S \cup b'$ generate V.*
         5. *Given a basis B for V and a linearly independent subset X of V, there is a subset b' of B such that $X \cup b'$ is a basis for V.*
      ii. Axiom of Choice for finite sets.
      iii. FSn: A choice-like statement that for any collection of sets there is a function assigning to each set in the family a *finite subset* of that set, rather than a single element of the set.
      iv. Choice for finite sets and FSn together in some sense make up the full axiom of choice.
      v. Introduce the vector space $L(X,F)$ and its' subspace $L_0(X, F)$, along with a basis for L_0, $B_x = \{x - y : y \in X \setminus \{x\}\}$.
         1. Prove B_x is a basis for L_0.
      vi. Introduce the weak product of a family of vector spaces over the field F, from which Bleicher extracts a choice function.

   b. Results for infinite dimensional vector spaces from Bleicher's paper.
      i. Lemma 1: . *If for some field F in which $1 + 1 \neq 0$, (i.e., not of characteristic 2) generating subsets contain bases for all vector spaces over F, then the axiom of choice for finite sets holds.*
      ii. Sketch of proof:
         1. Given a family of finite sets, form a disjoint family and construct the cross product of L_0 spaces over F for each set. Each space in

the cross product has a basis B_x. Each L_0 space has dimension n - 1, where n is the order of the finite set in the original family.

2. Construct a function π into each finite set based on how many times each element is represented in the basis B_x. Show this map yields strictly smaller subsets of each finite space.

3. Iterate to get a choice function, taking care of some details which arise in dealing with the 2 element sets.

4. Prove that in fact the result holds for fields of characteristic 2 as well. (Corollary)

iii. Lemma 2: *If for every vector space over some field F, proposition 3 holds, then FS1 holds.*

iv. Lemma 3 - extends Lemma 2 to FSp where p is prime, for fields of characteristic p.

v. Lemma 4 - Proposition 5 implies the axiom of choice for finite sets. This entails a different way to get a choice function.

vi. Theorem 1: (a. Choice) $\Leftrightarrow$ (b. #1 and 5) $\Leftrightarrow$ (c. #3 and 5). The main choice result here is that $c \Rightarrow a$, which follows from lemmas 2 and 4.

vii. Theorem 2: Choice $\Leftrightarrow$ Proposition 2 and 3

viii. Theorem 3: Choice $\Leftrightarrow$ Proposition 4

c. Summary of results so far.

4. Results for generalized vector spaces from Bleicher's paper.
   a. Introductory material
      i. Define generalized vector spaces, dependence and independence, and bases for generalized vector spaces. Note that 1 - 5 above hold in generalized vector spaces.
   b. Theorem 4: *Every generalized vector space has a basis $\Rightarrow$ Axiom of Choice.*
   c. Theorem 5: *Proposition 3 for generalized vector spaces $\Rightarrow$ Axiom of Choice.* (Prop. 3 is the disjoint subspace one)

5. Summary of Bleicher's results -
   a. Which theorems are equivalent to which strength of choice?
   b. Which propositions are independent in ZF?
   c. Why is it that one of lemma 1 or 4 may be weaker than it needs to be?
   d. How are choice functions constructed?

6. Tychonoff theorems equivalent to choice.
   a. Introductory material
      i. Briefly define topology and topological spaces, closed and open.
      ii. Discuss open and closed characterizations of compact.
      iii. Product topology.

   b. Tychonoff $\Rightarrow$ Choice (Smith/Kelley) Proof:

     i.   Get an indexed collection of non-empty sets. (X_a, $a \in A$), adjoin an x not in the union ($Y_a = \{x\} \cup X_a$). ? *Discuss: such x exists because we don't allow a universal set?*

    ii. Define a topology on each set in the family consisting of the empty set, and the complements of finite sets (i.e. cofinite sets).
        1. Prove this is a topology, for which each set is compact.

   iii. Take the cross product of the family. ($\mathsf{X} Y_a$) and introduce potential choice functions. ($Z_a = \{f : A \to \cup \, Y_a \, , \, f_a \in X_a \}$)

   iv. Show the family Z_a meets the hypothesis of Tychonoff and apply the theorem to get a choice function.

    v. Discuss the difference between finite choice for possibly infinite sets (used here) and AoC. *Kelley defines finite as a set which may be ordered so that every non-void subset has both a first and last element in the ordering.*

  c. Choice $\Rightarrow$ Tychonoff
     i. Kelley (using Alexander's Theorem)
        1. Proof of Alexander's Theorem showing the first of two instances of choice (?*Choice for finite sets, or just choice?*)
        2. Proof of Tychonoff using Alexander's theorem and showing the second instance of choice (*Should need the full axiom here)*
    ii. Bourbaki (via Kelley, not using Alexander's Theorem)
   iii. Holsztynski (using the Kuratowski-Zorn theorem)

  d. Concluding remarks for the Tychonoff proofs.

  e. Tychonoff for Hausdorff spaces $\Leftrightarrow$ Axiom of Choice for finite sets.
     i. Define Hausdorff spaces.
    ii. Alexander's Theorem $\Leftrightarrow$ Tychonoff for Hausdorff spaces. (*From Rubin and Scott, a handwritten note at the end of Smiths typed paper "On Certain Propositions...". Also mentioned by Kelley) This is interesting because Alexander's Theorem is used in Kelley's proof of Tychonoff. By the above, Choice for Finite sets is used to get to Alexander's Thm. There's another unavoidable instance of Choice in the proof of Tychonoff.*

7. Concluding remarks.

8. Bibliography

    Bleicher, M. N., *Some theorems on vector spaces and the axiom of choice*
    Holsztynski, W., *A Proof of Tychonoff Theorem*, 1972
    Kelley, John L., *General Topology* 1955
    Moore, Gregory H., *Zermelo's axiom of choice*. 1982
    Smith, James T. (?*Look up how to reference Dr. S's notes)*

Suppes (?*add reference*)