" Least-surface-area problem of a honeycomb "

It is interesting to note that polygons and polyhedra can be found in nature. With the help of electron microscope, one find icosahedral symmetry in some virous, or a re-gular polyhedra in a radiolaria which is a tiny sea creature. One may also find the regular hexagon in bees' honeycomb. The honeyconb of the bees consist of bee-cells which are prismatic vassels. The openings of the cells are regular hexagons and their bottoms consist of three equal rombi. The ancient Greek mathematician Pappus said " though God has given to men, most excellent Megathon, the best and most perfect understanding of wisdom and mathematics, He has allotted a partial share to some of the unreasoning creatures as well...This instinct may be observed to exist in other species of creatures, but it is specially marked amoung bees." Pappus goes on to conclude that bees construct their honeycomb in the most economical way.

Accepting the hypothesis that the honeycomb is build most economically, this paper examine the structure of the honeycomb in terms of its surface area.

L. FEJES TOTH defined a honeycomb as a set of congruent convix polyhedra, called cells, filling the space between parallel planes without overlapping and without interstices in such a way that (1) each cell has a face, called base or opening on one of the two planes but does not have faces on both planes (2) in the congrence of two cells their bases correspond to each other.

The distance between the parallel plane is the width of the honeycombs.

L. FEJES TOTH formulated two problems using his defina -tion of a honeycomb. They are (1) First isoperimetric problem for honeycombs. Amoung the polyhedra of volume v generating a honeycomb of width w find that one of leaset surface-area. (2) Second isoperimetric problem for honey-combs. Among the open cells of volume v generating a honeycomb (of any width) find that one of least surface-area.

In both cases, the bees'choice of hexagon as a base prove to be the best choice. In terms of plane tessella-tion, their is one of three regular tessellations. The simplest regular tessellation is the one with equala-teral triangles (Fig. 1) and then the one with squares (Fig. 2) and the last, with regular hexagons (Fig.3). Suppose an equilateral triangle, a square, and a hexagon, all have the same area A, and each has a side α, β, γ respectively (Fig. 4-6).

The perimeter of the thrangle is 3α and the area A in terms of α is $3/4\alpha^2$.

Since $$(\tfrac{1}{2})^2 + h^2 = \alpha^2$$

$$h^2 = \alpha^2 - (\tfrac{1}{2})^2$$

$$h = \sqrt{\alpha^2 - (\tfrac{1}{2})^2}$$

$$h = \sqrt{3}/2\,\alpha$$

$$\left(\tfrac{1}{2}\right)\cdot\left(\alpha\cdot\sqrt{3}/2\,\alpha\right) = A$$

$$\sqrt{3}/4\,\alpha^2 = A$$

The perimeter of the square is 4β, and in terms of α, it is $2\cdot\sqrt[4]{3}\,\alpha$ since

$$(\beta)^2 = A$$

$$\beta^2 = \sqrt{3}/4\cdot\alpha^2$$

$$\beta = \sqrt{\sqrt{3}/4\cdot\alpha^2}$$

$$\beta = \sqrt[4]{3}/2\cdot\alpha$$

$$4\beta = 4\cdot\sqrt[4]{3}/2 = 2\cdot\sqrt[4]{3} = 2.63\alpha$$

The perimeter of the hexagon is 6γ, and in terms of α, it is $\sqrt{6}\,\alpha$ since

$$\left(\tfrac{1}{2}\gamma\right)^2 + \eta^2 = \gamma^2$$

$$\eta^2 = \gamma^2 - \left(\tfrac{1}{2}\gamma\right)^2$$

$$\eta = \sqrt{3}/2\,\gamma$$

$$\tfrac{1}{2}\gamma\cdot\sqrt{3}/2\cdot\gamma = A/6$$

$$\sqrt{3}/4\cdot\gamma^2 = 1/6\cdot\sqrt{3}/4\cdot\alpha^2$$

$$\gamma^2 = 1/6\,\alpha^2$$

$$\gamma = 1/\sqrt{6}\,\alpha$$

$$6\gamma = \sqrt{6}\,\alpha \doteq 2.45\alpha$$

Thus, given the area A,

The perimeter of an equilateral triangle $>$ the perimeter of a square $>$ the perimeter of a hexagon

So the hexagon is the best choice and the bees save their wax most.

It seems that bees save more wax if they use a polygon with more sides than six for a base. However, it is impossible to obtain a regular tessellation with polygons with more sides than six. For example, a plane tessellation with regular octagons is a semiregular tesselation with squares(Fig. 7). The perimeter of a octagon and four squares would be larger than a hexagon. The bees would waste their wax of they use this tessellation.

In second isometric problem for a honeycomb, the problem is how to fill space in such a way that the surface area of each solid would be minimized. Closed-packing systems of polyhedra or isometric space-filling polyhedra are analogous to the plane tessellations. An isometric space-filling polyhedron with its duplicates fill space without leaving any space in between. The only regular (platonic) and Archimedean polyhedra that can be used for isometric space filling are cubes and truncated octahedra (Fig. 8) respectively. Rhombic dodecahedra (Fig. 9), being the Archimedian duals (from dualizing the cuboctahedra), also fill space completely.

Now, when two bases of honeycombs are put together, a "telescpocially elongated" rombic dodecahedron is obtained. So the bees' choice of a cell for their honeycombs seems convincing. However, when a truncated octahedron is cut into half by the plane perpendicular to one of its hexagonal faces, a cell generating a honeycombs are obtained., too.

Suppose a truncated octahedron and a rhombic dodecahedron having the same volume V, then the total surface area of the truncated octahedron would be smaller than the total surface area of the rhombic dodecahedron. So the truncated octahedron half is the better shape in terms of minimizing the surface area. In reality the bees don't build their cells with the oprimal height. They build rather deep cells. This fact leads to consider First isometric problem for a honeycomb.

For the solution to First isometric problem of a honeycomb, L. FEJES TOTH introduces a polyhedra obtained as follow : 1. Elongate the vertical diagonal of a regular octahedron symmetrically in both directions so as to obtain an octahedron with dihedra angles of 120° . Truncate this by two horizontal planes touching the insphere of the body. 3. Cut off the remaining corners of the octahedron by planes perpendicular to the corresponding diagonals of the octahedran. 4. Perform 3 at a suitable equal depth so the hexagonal faces will be centro-symmetric. The result obtained is a parallelohedron (Fig. 10). It consists of two squares for which each square is surrounded by hexagons and four rhombi. Let the length of the square be s then the length p parallel to s will be 3s/2 (Fig. 11).

The length of the rhombi and the diagonals can be calculated as follow, Fig. 12 is a partial drawing of the solid, the coloured solid, is truncated part. From Fig.12 the length of the rhombi is $\sqrt{(s/2)^2 + (s/4)^2} = \sqrt{5} \cdot s/4$.

, the shorter diagonal is $\sqrt{2}/2s$ and the longer diagonal is $\sqrt{3}/2 \cdot s$.

The total surface area of the solid is

$$s = 2s^2 + 8 \cdot 5s^2/4 = 4 \cdot \sqrt{6}/8 \cdot s^2 = (24 + \sqrt{6})\ \frac{s^2}{2}$$

The volume of the solid (the volume of regular prism with altitude 3s/2)

$$V = \sqrt{3} \cdot 9 \cdot s^3/4$$

To compare the surface area of the bottom, the solid is elongated in the axial direction of one of its regular zones by x. Set s=1, simplify the surface area of the new solid would be

$$S_x = (24 + \sqrt{6})/2 + 6x$$

and its volume would be

$$V_x + 9 \cdot \sqrt{3}/4 + 3 \cdot \sqrt{3} \cdot x / 2$$

Let V_x be the volume of a rhombic dodecahedron with a regular hexagonal section side-length 1 :

$$\frac{9\sqrt{3}}{4} + \frac{3\sqrt{3}}{2}x = 3\sqrt{\frac{3}{2}}$$

$$3\sqrt{\frac{3}{2}}\,x = 3\sqrt{\frac{3}{2}} - \frac{9\sqrt{3}}{-4}$$

$$x = \frac{-3-\sqrt{8}}{2}$$

$$S_x = \frac{24 + /6}{2} + 6 \left(- \frac{3 - /8}{2} \right)$$

$$= \frac{24 - 18 + /6 + 6/8}{2}$$

$$= 3 + \frac{/2 \ (/3 + 12)}{2}$$

$$= 12.71$$

$S_x <$ The surface area of the rhombic dodecahedron which has the same value

Cells generating a honeycomb is obtained from the rhombic dodecahedron and the shortened snub betahedron.

So the cells generating the honeycomb in nature is not the most economical one, but the one with the base consist of two hexagons and two rhombi.

Even so, the bees only save their wax .35% of their area of the opening of the cell. The artificial one would be more complicated to construct. After all, What Pappus said is not correct thoretically, the bees deserve his words.

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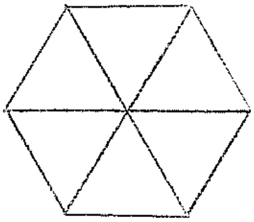

Fig. 1.

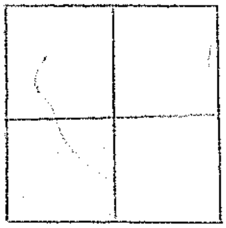

Fig. 2.

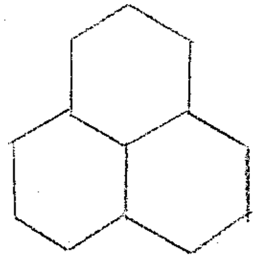

Fig. 3

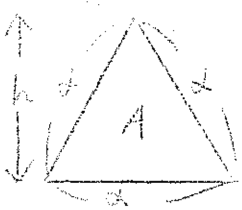

Fig. 4

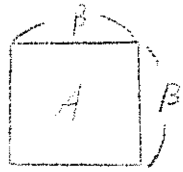

Fig. 5

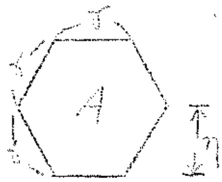

Fig. 6

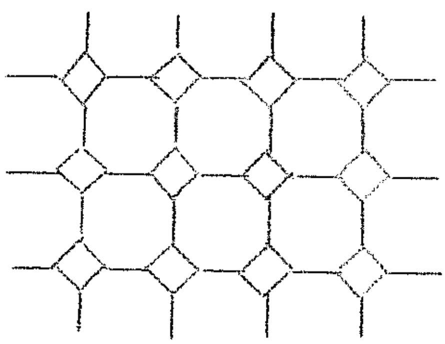

Fig. 7

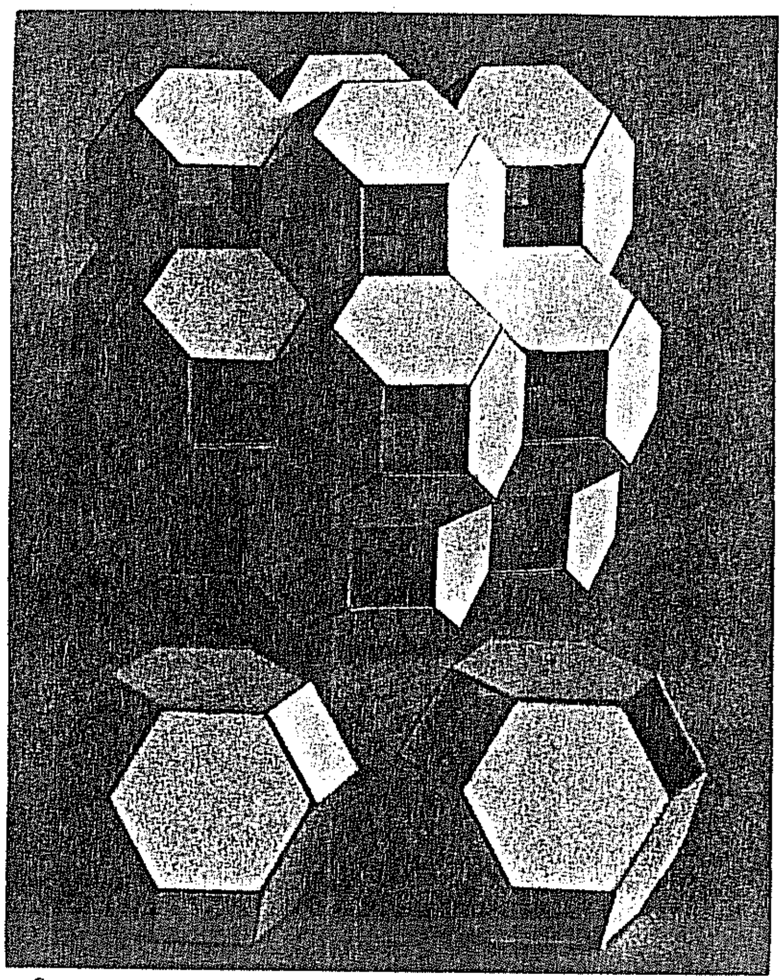

Fig. 8

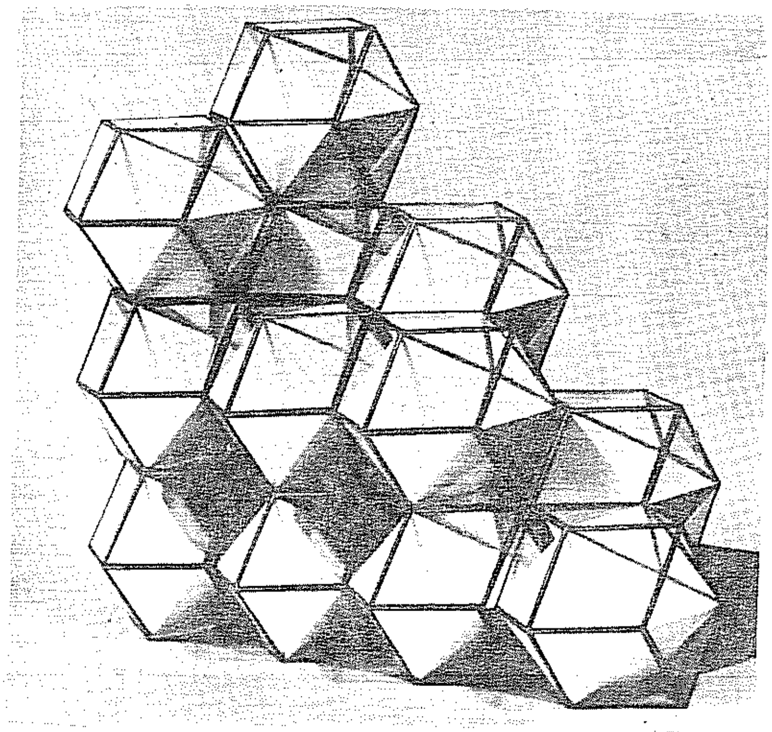

Fig. 9

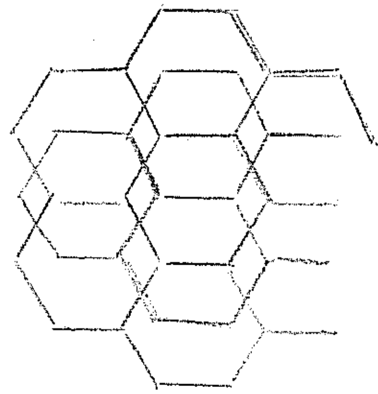

Fig. 11

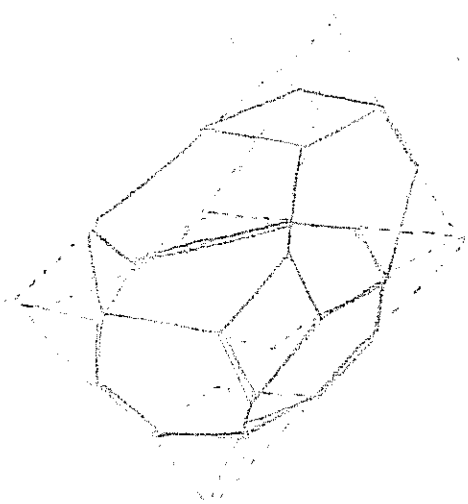

Fig. 10

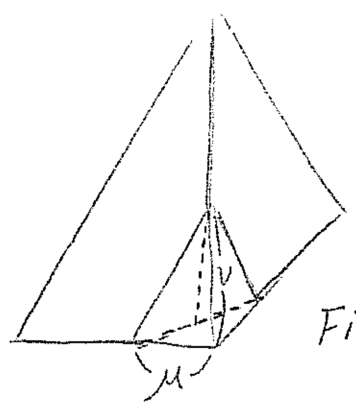

Fig 12