VIRTUAL CLASSES

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Many mathematical theories utilize classes composed of their basic objects. What set-theoretic[1] assumptions does that require? In fact, many elementary manipulations with such classes do not rely on any basic principles specifically about classes, because part of set theory really consists of logical principles clad in special garb for mathematical discourse. These notes first show how this can be done rigorously in the context of elementary logic. New notation is introduced, whose meaning is defined through elementary logic. The corresponding manipulation rules are based not on any specific mathematics, but on the general underlying logic. Classes *seem* to be in use, but really *aren't*; so this is called the *virtual*-class technique.[2]

Since virtual-class manipulation is a linguistic, not mathematical, technique, it's best to describe it in terms of a formal language. Suppose you're presenting a mathematical theory in an elementary language. Given any formula Φ, you can introduce the symbolism $\{x : \Phi\}$, called an *abstraction*. It's read, *the class*[3] *of all* x *such that* Φ. Even though this *looks* like ordinary set notation, don't manipulate it by itself. Instead, use it only in contexts like $\{x : \Phi\} = \{x : \Psi\}$, where Ψ is another such formula, or $t \in \{x : \Phi\}$ for any term t, provided no occurrence of x in Φ lies within the scope of a quantifier governing any variable free in t. Now, regard these new formulas as abbreviations:

$\{x : \Phi\} = \{x : \Psi\}$	for	$\forall x[\Phi \Leftrightarrow \Psi]$
$t \in \{x : \Phi\}$	for	the result of substituting t for all free occurrences of x in Φ.

With care, you can manipulate an abstraction $\{x : \Phi\}$ *almost* like a term; you must regard all its occurrences of x as bound. *Caution: these* rules *do not* justify use of expressions of the form $t = \{x : \Phi\}$ or $\{x : \Phi\} \in t$.

You can define some *finite* virtual classes of given objects by abstraction: for example,

$\phi = \{x : x \neq x\}$	—the *empty* class
$\{a\} = \{x : x = a\}$	—a *singleton*
$\{a,b\} = \{x : x = a \vee x = b\}$	—an unordered *pair*

[1] In this paragraph, as in most mathematical usage, *set* and *class* are regarded as synonymous. That will soon change.

[2] *Virtual* is the opposite of *transparent*. What is transparent is really there but seems not to be. What is virtual seems to be there but really isn't.

[3] Henceforth, the word *class* is used for this notion only.

The first of these examples should suggest that you can define the *universal* class, too:

$$V = \{ x : x = x \} .$$

Although that would lead to difficulty in *set theory*, it doesn't here, because V is a virtual class. Speaking about V is merely speaking about the formula $x = x$, which is valid for all x in the universe of discourse of the elementary theory under consideration. Thus V simply provides a way of speaking about that universe of discourse.

Suppose virtual classes A and B correspond to formulas Φ and Ψ. Then you can define an abbreviation

$$A \subseteq B \qquad \text{for} \qquad \forall x [x \in A \Rightarrow x \in B] .$$

The first expression is the familar *subclass* relationship. The latter expression in turn stands for $\forall x [\Phi \Rightarrow \Psi]$. You can then prove the following result from logical principles alone:

Theorem 1. $\phi \subseteq A \subseteq V$ for every class A.

In considering such results, you must realize that A is a variable ranging not over individuals of the theory under consideration, but over virtual classes of them, and thus it really ranges over formulas of the language in which that theory is expressed. It is a *linguistic* variable, a feature of the *metatheory*. Elementary logic provides no way to quantify over such a variable.

You can use the same technique for defining the familiar Boolean operations on virtual classes. For example, if virtual classes A and B correspond to certain formulas, you can define their *union* as an abbreviation:

$$A \cup B \qquad \text{for} \qquad \{ x : x \in A \vee x \in B \} .$$

And you can prove familiar results from logical principles alone. As an example, for any object x,

$$x \in A \cup B \Leftrightarrow x \in A \vee x \in B \Leftrightarrow x \in B \vee x \in A \Leftrightarrow x \in B \cup A$$

because the middle equivalence is tautologous. Therefore, $A \cup B = B \cup A$, the commutative law.

When the formula Φ is a conjunction whose first clause is a condition on x, such as $x \in A$, we often abbreviate the corresponding abstraction slightly, as for the *difference* of A and B:

$$A - B = \{ x \in A : x \notin B \} = \{ x : x \in A \ \& \ x \notin B \} .$$

Use of nested abstractions can lead to a problem familiar to software designers: unless some care is taken with the definitions, an expression involving several abstractions could conceivably be parsed in more than one way, leading to expressions in the formal language that are not equivalent. Having worked in that field, I know how to avoid such problems, but I won't present any details here. You may consult some good text on programming language design.

Virtual classes will be used later to study how much set theory can be encompassed in an elementary theory with few and weak axioms. Only when you realize what theorems result from the logic alone can you assess the consequences of various axioms. The elementary language used here for formal set theory has just one nonlogical symbol $\in$, intended to stand for the membership relation. Its universe of discourse is intended to consist entirely of sets. Virtual-class techniques will allow us to appear to be speaking of some sets, but those expressions constructed that way can be translated into the basic language with just the $\in$ relation.

Some texts regard the universe of discourse of such a set theory as also encompassing individuals that are not sets but can belong to sets. Those are often called *Urelemente*.[4] Other texts extend the language of set theory formally to admit construction of abstractions like those we use to indicate virtual classes, and allow some, but not all, manipulations of them as though they were terms of an elementary language. The corresponding universe of discourse must encompass those objects, too, and the axioms of the theory must provide a way to determine which of them are sets. These features all introduce clutter, and are not necessary for using set theory to do ordinary mathematics. These notes, therefore, do not employ them.

How should virtual-class techniques be used to interpret expressions such as

$$t = \{x : \Phi\} \qquad \{x : \Phi\} \in u,$$

where Φ is a formula of the formal language of set theory, t and u are variables, and t does not occur in Φ? These expressions are simply abbreviations,

$$t = \{x : \Phi\} \quad \text{for} \quad \forall x[x \in t \Leftrightarrow \Phi]$$
$$\{x : \Phi\} \in t \quad \text{for} \quad (\exists s \in t)[s = \{x : \Phi\}].$$

To make the first of these conventions compatible with that governing expressions such as $\{x : \Phi\} = \{x : \Psi\}$, the formal set theory should incorporate the *axiom of extensionality*:

$$(\forall x,y)[\forall w[w \in x \Leftrightarrow w \in y] \Rightarrow x = y].$$[5]

The now-standard definition of the ordered pair of two objects x,y uses this technique:

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[4] In German, the particle *Ur*- often means *primitive* or *original*. *Pilsener Urquelle* is beer from an original source near Pilsen.

[5] $(\forall x,y)[x = y \Rightarrow \forall w[w \in x \Leftrightarrow w \in y]]$ is deducible from *logical* axioms. Some presentations of first-order logic, however, do not include a theory of equality: they require each application to define its own equality. If such a logic were in use here, $x = y$ would be defined as $\forall w[w \in x \Leftrightarrow w \in y]$.

$$<x,y> = \{\{x\},\{x,y\}\}.$$

Proof of the following theorem[6] is a routine, though messy, exercise:

Theorem 2. In the formal set theory the *axiom of pairs*,

$$(\forall x,y)\exists z[w \in z \Leftrightarrow w = x \vee w = y],$$

implies

$$(\forall v,w,x,y)[<v,w> = <x,y> \Leftrightarrow v = x \;\&\; w = y].$$

Routine exercises

1. Use virtual-class techniques to prove theorem 2.

2. Invent virtual-class definitions for the union and intersection of a set. Then prove the following results, keeping track of whatever set-theoretic axioms are needed:
   a. $\bigcup\bigcup<x,y> = x \cup y$
   b. $\bigcup\bigcap<x,y> = x$
   c. $\bigcup\bigcup<x,y> - \bigcup\bigcap<x,y> = y$. (*Yikes*!)

   Those could be called unintended side-effects of the Kuratowski definition of ordered pair. How can they be used to clean up the proof of theorem 2?

3. Use virtual-class techniques for determining when $\{a, b, c\} \cap \{b, c, d\} = \{b, c\}$.

[6] Due to Kuratowski 1921.