PARTIALLY ORDERED SETS

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A reflexive transitive relation on a nonempty set X is called a *quasi-ordering* of X. An ordered pair $<X, \le>$ consisting of a nonempty set X with a quasi-ordering $\le$ is called a *quasi-ordered set*. If $\le$ is also *weakly antisymmetric*—that is, for all $x, y \in X$

$$x \le y \;\;\&\;\; y \le x \;\Rightarrow\; x = y,$$

then $\le$ is called a *partial* ordering and $<X, \le>$ is a *partially ordered set*. Clearly, the identity relation is the smallest quasi-ordering on X. Every equivalence on X is a quasi-ordering, but the identity is the only one that's also a partial ordering. The familiar order relations $\le$ on various sets of numbers are partial orderings. The inclusion relation $\subseteq$ is a partial ordering on any family of sets.

A quasi- or partial ordering $\le$ of a nonempty set X induces a quasi- or partial ordering $\le_S = \le \cap (S \times S)$ of any nonempty subset $S \subseteq X$. $<S, \le_S>$ is called a *substructure* of $<X, \le>$.

A *homomorphism* from a quasi-ordered set $<X, \le>$ to a quasi-ordered set $<X', \le'>$ is a function $\varphi : X \to X'$ such that for all $x, y \in X$,

$$x \le y \;\Rightarrow\; \varphi(x) \le' \varphi(y).$$

For example, the function that assigns to each real number t the largest integer $\le t$ is a homomorphism from $<\mathbb{R}, \le>$ to $<\mathbb{Z}, \le>$.

You can construct a homomorphism φ from any quasi-ordered set $<X, \le>$ to a closely related partially ordered set, as follows. According to substantial problem 2 in the *Equivalence relations* notes, the relation $E = \le \cap \breve{\le}$ is an equivalence on X. If $x\,E\,x'$ and $y\,E\,y'$, then $x \le y$ if and only if $x' \le y'$. Thus, you can define a relation $\le/E$ on X/E by setting, for any $x, y \in X$,

$$x/E \le/E\; y/E \;\Leftrightarrow\; x \le y\,.$$

You can show easily that $\le/E$ partially orders X/E and the quotient map $x \mapsto x/E$ is a homomorphism from $<X, \le>$ to $<X/E, \le/E>$. This result, due to Schröder [1890–1905] 1966, volume 1, shows that the theory of quasi-ordered sets isn't very interesting in itself. But it is used in the theory of cardinality.

An *isomorphism* from a quasi-ordered set $<X, \le>$ to a quasi-ordered set $<X', \le'>$ is a bijection $\varphi : X \to X'$ such that for all $x, y \in X$,

$$x \le y \;\Leftrightarrow\; \varphi(x) \le' \varphi(y)\,.$$

For example, the exponential function is an isomorphism from $<\mathbb{R}, \leq>$ to $<(0,\infty), \leq>$.

If $<X, \leq>$ is a partially ordered set and $x \in X$, define

$$x{\downarrow} = \{t \in X : t \leq x\} \qquad x{\uparrow} = \{t \in X : x \leq t\}.$$

The functions $x \to x{\downarrow}$ and $x \to x{\uparrow}$ are isomorphisms from $<X, \leq>$ to substructures of $<\mathscr{P}X, \subseteq>$ and $<\mathscr{P}X, \supseteq>$. You need only the reflexivity and transitivity of $\leq$ to prove that they are homomorphisms, but you must use its weak antisymmetry to show that they are injective. Because of this result, due to Garrett Birkhoff, the theory of partially ordered sets loses some interest—you can often use the algebra of sets instead. Nevertheless, this theory serves as a very convenient basis for many others.

The identity I_X is an isomorphism from any quasi-ordered set $<X, \leq>$ to itself. If φ is an isomorphism from $<X, \leq>$ to a quasi-ordered set $<X', \leq'>$, then φ^{-1} is an isomorphism from $<X', \leq>$ to $<X, \leq>$. If φ is an isomorphism from $<X, \leq>$ to $<X', \leq'>$ and χ is one from $<X', \leq>$ to a quasi-ordered set $<X'', \leq''>$, then $\chi \circ \varphi$ is an isomorphism from $<X, \leq>$ to $<X'', \leq''>$.

An isomorphism from a quasi-ordered set $<X, \leq>$ to itself is called an *automorphism* of $<X, \leq>$. These automorphisms form a subgroup of the symmetric group on X, called the *automorphism group* of $<X, \leq>$.

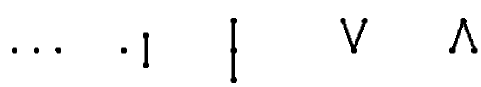
Two quasi-ordered sets are called *isomorphic* if there exists an isomorphism from one to the other. Any quasi-ordered set isomorphic to a partially ordered set is itself partially ordered. The relation "isomorphic" is an equivalence on any family of quasi-ordered sets; equivalence classes under this relation are called *isomorphism classes*.

The converse $\gtrsim$ of a quasi- or partial ordering $\leq$ of a nonempty set X is itself a quasi- or partial ordering of X; the partially ordered set $<X, \gtrsim>$ is called the *dual* of $<X, \leq>$. A quasi-ordered set is called *self-dual* if it's isomorphic to its dual.

Isomorphism classes of finite partially ordered sets are nicely described by their *Hasse diagrams*:

one element

two elements 

three elements 

four elements sixteen different diagrams

Fourteen of these diagrams are self dual.

An element x of a partially ordered set $<X, \leq>$ is called *maximal* if for each $y \in X$,

$$y \in X \ \& \ x \leq y \ \Rightarrow \ x = y.$$

A *maximum* element is an element x such that for each $y \in X$,

$$y \in X \ \Rightarrow \ y \leq x.$$

You can show easily that $<X, \leq>$ has at most one maximum element and that any maximum element is maximal. Similar definitions and results hold for *minimal* and *minimum* elements. Every finite partially ordered set obviously has one or more maximal and one or more minimal elements.

A partial ordering $\leq$ of a nonempty set X is called *linear* if it satisfies the *dichotomy law*: for any $x, y \in X$,

$$x \leq y \ \vee \ y \leq x;$$

in that case $<X, \leq>$ is called a *linearly* ordered set. The familiar orderings of various sets of numbers are linear. Every nonempty subset and every homomorphic image of a linearly ordered set is linearly ordered. For a linearly ordered set, the notions "maximal" and "maximum" coincide, as do "minimal" and "minimum"; thus any finite linearly ordered set has maximum and minimum elements. Any two finite linearly ordered sets with the same number of elements are isomorphic, hence every finite linearly ordered set is self dual.

Routine Exercises

1. A *strict ordering* of a set X is a binary relation $<$ on X that satisfies the *antireflexivity*, *strong antisymmetry*, and *transitivity* laws: for all $x, y, z \in X$,

   $$\neg\, [\, x < x \,] \qquad \neg\, [\, x < y \ \& \ y < x \,] \qquad x < y \ \& \ y < z \ \Rightarrow \ x < z.$$

   Show that in this definition the strong antisymmetry requirement is redundant. If R is a partial ordering of X, define

   $$x\, R^s\, y \ \Leftrightarrow \ x\, R\, y \ \& \ x \neq y$$

   for all $x, y \in X$. Show that R^s strictly orders X. If R is a strict ordering of X, define

   $$R^p = R \cup I_X.$$

   Show that R^p partially orders X. Show that $R^{sp} = R$ for each partial ordering R and $R^{ps} = R$ for each strict ordering R. These results show that the notions "partial ordering" and "strict ordering" are just two aspects of the same concept.

2. Let I be a nonempty set and $<X, \le>$ be a quasi-ordered set. Define a relation R on X^I by setting, for each $f, g \in X^I$,

$$f\,R\,g \;\Leftrightarrow\; \forall i\,[\,f_i \le g_i\,]\,.$$

Show that R quasi-orders X^I. Show that if $\le$ is a partial ordering, then so is R. Find an example where $\le$ is linear but R isn't.

3. Show that the intersection of a nonempty family of quasi-orderings of X quasi-orders X. Let $\mathcal{Q}$ be a nonempty family of quasi-orderings of X such that for any $R, S \in \mathcal{Q}$ there exists $T \in \mathcal{Q}$ with $R, S \subseteq T$. Show that $\bigcup\mathcal{Q}$ quasi-orders X. Do the same for partial orderings. What about linear orderings?

Substantial problems

1. How many isomorphism classes of five-element partially ordered sets are there? Which are self dual?

2. Let $<X_0, \le_0>$ and $<X_1, \le_1>$ be partially ordered sets. Define

$$X = (X_0 \times \{0\}) \cup (X_1 \times \{1\})\,.$$

Show that X is partially ordered by the relation $\le$ defined by setting, for all $x_0, y_0 \in X_0$ and $x_1, y_1 \in X_1$,

$$<x_0, 0> \le <y_0, 0> \;\Leftrightarrow\; x_0 \le_0 y_0 \qquad \neg\,[\,<x_0, 0> \le <x_1, 1>\,]$$
$$<x_1, 1> \le <y_1, 1> \;\Leftrightarrow\; x_1 \le_1 y_1 \qquad \neg\,[\,<x_1, 1> \le <x_0, 0>\,]\,.$$

Show that $\le$ partially orders X. The partially ordered set $<X, \le>$ is called the *cardinal sum* $<X_0, \le_0> + <X_1, \le_1>$. Let $\mathcal{X}_0$, $\mathcal{X}_1$, and $\mathcal{X}_2$ be partially ordered sets. Show that

- $\mathcal{X}_0 + \mathcal{X}_1$ is isomorphic to $\mathcal{X}_1 + \mathcal{X}_0$
- $(\mathcal{X}_0 + \mathcal{X}_1) + \mathcal{X}_2$ is isomorphic to $\mathcal{X}_0 + (\mathcal{X}_1 + \mathcal{X}_2)$
- the dual of $\mathcal{X}_0 + \mathcal{X}_1$ is isomorphic to the cardinal sum of the duals of $\mathcal{X}_0$ and $\mathcal{X}_1$.

You could call the first two of these results *commutative* and *associative* laws. For what kinds of partially ordered sets can you derive the *cancellative* law,

- if $\mathcal{X}_0 + \mathcal{X}_1$ is isomorphic to $\mathcal{X}_0 + \mathcal{X}_2$ then $\mathcal{X}_1$ is isomorphic to $\mathcal{X}_2$?

3. Same as substantial problem 2, for *cardinal products* $<X, \le> = <X_0, \le_0> \times <X_1, \le_1>$, where $X = X_0 \times X_1$ and for all $x_0, y_0 \in X_0$ and $x_1, y_1 \in X_1$,

$$<x_0, x_1> \le <y_0, y_1> \Leftrightarrow x_0 \le_0 y_0 \;\&\; x_1 \le_1 y_1 .$$

Cardinal sums and products were introduced in Whitehead and Russell 1910 and studied further in Birkhoff 1937.

4. Let G be a group; use multiplicative notation for its operations. Define a binary relation $\le$ on G by setting, for all $x, y \in G$,

$$x \le y \Leftrightarrow \exists z\,[\,x = yz\,] .$$

Show that $\le$ quasi-orders G. Show that for each $g \in G$ the function $\varphi_g : x \to gx$ is an automorphism of $<G, \le>$. Show that the function $g \to \varphi_g$ is a group isomorphism from G to a subgroup of the automorphism group of $<G, \le>$. Thus, every group is isomorphic to a subgroup of the automorphism group of some quasi-ordered set.

Project

Pursue substantial problems 2 and 3 further. Define the cardinal sum and product of families of (arbitrarily many) partially ordered sets. Define the concept of cardinal power. Consider ordinal sums, products, and powers and lexicographic sums and products in the same vein. Prove as many rules as you can.

References

Birkhoff 1948 is the bible of this area of mathematics. See also the third edition, 1967, which differs considerably.
Kurosh 1963, 20–22.