NATURAL NUMBERS

James T. Smith
San Francisco State University

These notes introduce some aspects of the theory of natural numbers, and provide some guidelines for further study.

A *Peano system* is an algebraic structure $<N,^+,0>$, where N is a set, $^+$ is an operator on N, $0 \in N$, and the following *postulates* are satisfied:

$(\forall n \in N)[0 \neq n^+]$	—*zero*
$(\forall m,n \in N)[m^+ = n^+ \Rightarrow m = n]$	—*injectivity*
$(\forall S \subseteq N)\big[0 \in S \ \&\ (\forall n \in S)[n^+ \in S] \Rightarrow S = N\big]$	—*recursion*

Terms of the form n^+ are intended to suggest and behave like corresponding terms $n + 1$ in natural-number arithmetic, and n^+ is called the *successor* of n. Richard Dedekind and Giuseppe Peano showed in [1888] 1963 and [1889] 1973, respectively, that the order relation and the binary addition, multiplication, and exponentiation operators can be defined in such a structure, and their standard properties derived, using set theory like that presented in the *Basic Set Theory* notes.[1] Dedekind also showed that any two Peano systems $<N,^+,0>$ and $<N',^*,o>$ are *isomorphic*: there exists a bijection $n \to n'$ from N to N' such that $0' = o$ and $(m^+)' = (m')^*$ for all $m \in N$. Thus *any* convenient Peano system can be used to represent our intuitive concept of the natural-number system.

The principal techniques used in such arguments are recursive definition and proof. The latter has already been stated as a defining property of Peano systems. The former is explained and justified by a theorem to follow. Proving that theorem, however, requires some definitions and a brief analysis of the order relation.

Let $<N,^+,0>$ be a Peano system. A subset $S \subseteq N$ is called *closed* if

$$(\forall m \in N)[m' \in S \Rightarrow m \in S].$$

For example, the empty subset and N itself are closed. The *order relation* $\leq$ is defined by setting

$$m \leq n \Leftrightarrow (\forall \text{ closed } S \subseteq N)[n \in S \Rightarrow m \in S]$$

for any $m,n \in N$. For convenience, also set

[1] Dedekind's work was in preparation years before publication. Peano worked independently and more quickly. He saw Dedekind [1888] 1963 shortly before he completed [1889] 1973, and used it, with acknowledgment, to confirm his own results. Both of those works are readable today, though Peano's is ultraconcise and written almost entirely in his ideography, the source of today's logical notation. The term *Peano system* is my own. Surprisingly, there is no standard term for this type of structure.

$n\downarrow = \{m \in N : m \le n\}$
$m < n \Leftrightarrow m \le n \;\&\; m \ne n.$

Theorem 1. If $<N,^+,0>$ is a Peano system, then

a. $(\forall n \in N)[n \le n]$
b. $(\forall l,m,n \in N)[l \le m \;\&\; m \le n \Rightarrow l \le n]$
c. $(\forall n \in N)[n \le n^+]$
d. $(\forall n \in N)[n\downarrow$ is closed$]$
e. $0\downarrow = \{0\}$
f. $(\forall m,n \in N)[m < n^+ \Rightarrow m \le n]$ —lemma for part (h)
g. $(\forall n \in N)[n^+ \not\le n]$
h. $(\forall m,n \in N)[m < n^+ \Leftrightarrow m \le n]$.

Proof. Parts (a–d) are immediate consequences of the preceding definitions; part (e) stems from the zero postulate.

To prove part f, suppose $m < n^+$; it is to be shown that $m \le n$. To that end, consider any closed subset $S \subseteq N$ such that $n \in S$; it is to be shown that $m \in S$. Let $S' = S \cup \{n^+\}$, so that $n^+ \in S'$. Should S' be closed, it would follow from $m < n^+$ that $m \in S'$, and thus $m \in S$ as required, because $m \ne n^+$. Thus it remains only to show that S' is closed. Let $l \in N$ and $l^+ \in S'$; it must be shown that $l \in S'$. Since $l^+ \in S'$, either $l^+ \in S$ or $l^+ = n^+$. In the first case, $l \in S$ because S is closed; in the second, $l = n \in S$ by the injectivity postulate and the specification of S.

Part (g) will be proved recursively. The initial step is to show that $0^+ \not\le 0$; that follows from part e and the zero postulate. The recursive step is to assume $n \in N$ and $n^+ \not\le n$, then prove on that basis that $n^{++} \not\le n^+$. Were $n^{++} \le n^+$ and $n^{++} \ne n^+$, then $n^{++} \le n$ by part (f), and hence $n^+ \le n$ by part (c), which would contradict the assumption. Were $n^{++} = n^+$, then $n^+ = n$ by the injectivity postulate, and hence $n^+ \le n$, which again would contradict the assumption.

Part (h) follows immediately from parts (f) and (g). ♦

Many more properties of the successor operator and order relation can be derived at this stage. Those in theorem 1 were presented here because they are particularly simple or specifically required for the next result, which justifies recursive definition.[2]

Theorem 2. Let $<N,^+,0>$ be a Peano system, S be any set, $z \in S$, and $g : N \times S \to S$. Then there exists a unique $f : N \to S$ such that

$$f(0) = z \;\&\; (\forall n \in N)[f(n^+) = g(n,f(n))].$$

Proof. Define $f =$

$$\{<n,s> : \; n \in N \;\&\; (\exists e : n\downarrow \to S)[e(0) = z \;\&\; (\forall m < n)[e(m^+) = g(m,e(m))]]\}.$$

[2] Dedekind formulated and proved this theorem in [1888] 1963, section 132. In [1889] 1983 Peano didn't recognize the need for it.

It will be proved recursively that Domain $f = N$. The initial step is to show that $0 \in$ Domain f. That requires verifying

$$(\exists e : 0\downarrow \to S)\big[e(0) = z \;\&\; (\forall m < 0)[e(m^+) = g(m,e(m))]\big]:$$

the second conjunct is vacuously satisfied, so $e = \{<0,z>\}$ satisfies the condition. The recursive step is to assume that $m \in$ Domain f and prove on that basis that $m^+ \in$ Domain f. The assumption provides $e : m\downarrow \to S$ such that

$$e(0) = z \;\&\; (\forall l < m)\,[e(l^+) = g(l,e(l))]\big].$$

To verify that $m^+ \in$ Domain f consider the set

$$e' = e \cup \{<m^+, g(m,e(m)>\}.$$

By theorem 1g, $e' : n^+\downarrow \to S$ and

$$e'(0) = z \;\&\; (\forall m < n^+)[e(m^+) = g(m,e(m))]\big].$$

It remains to verify the uniqueness of f. To this end, suppose $f' : N \to S$ and

$$f'(0) = z \;\&\; (\forall n \in N)[f'(n^+) = g(n,f'(n))].$$

One shows that $f = f'$ by proving recursively that

$$(\forall n \in N)[f(n) = f'(n)].\blacklozenge$$

It will be useful later to recall that just twice in the preceding proofs were sets constructed by abstraction: in the clauses "let $S' = S \cup \{n^+\}$" and "consider the set $e' = e \cup \{<m^+, g(m,e(m)>\}$" in the proofs of theorems 1f and 2. Each instance required justification of the formation of the union of two given sets, and of the set consisting solely of one given member. Besides the apparent occurrences of such sets in those clauses, those processes also underlie the construction of ordered pairs.

Theorem 3. Between any two Peano systems $<N,{}^+,0>$ and $<S,{}^*,o>$ there is a unique isomorphism.

Proof. Define $g : N \times S \to S$ by setting $g = \{<<n,s>,s^*> : n \in N \;\&\; s \in S\}$ and use theorem 2 to define $f : N \to S$ recursively in terms of g, so that $f(0) = o$. Then prove recursively that

$$(\forall m \in N)\big[f(m^+) = f(m)^* \;\&\; (\forall n \in N)[f(m) = f(n) \;\Rightarrow\; m = n]\big].$$

Thus f is an isomorphism. To show that it's unique, suppose that $f' : N \to S$ bijectively, $f'(0) = o$, and $(\forall n \in N)[f'(x^+) = f'(x)^*]$, then prove recursively that $(\forall n \in N)[f(n) = f'(n)]$. $\blacklozenge$

You're invited to follow one of the standard texts—for example, Stoll [1963] 1979, chapter 2—to define the binary addition, multiplication, and exponentiation operators in a Peano system, and to prove their properties, which mimic the familiar arithmetic of natural numbers. You should find the interface between these notes and that presentation seamless, except for the treatment of order. The definition of order given here differs from the usual one; their equivalence is stated by

Theorem 4. If $<N, ^+, 0>$ is a Peano system, then for all $m, n \in N$

$$m \le n \iff (\forall T \subseteq N)[m \in T \ \& \ (\forall l \in T)[l^+ \in T] \Rightarrow n \in T].$$

Proof. Each direction of the equivalence is proved by contraposition, considering the set $S = N - T$, which is closed if and only if $(\forall l \in T)[l^+ \in T]$. ♦

These notes concentrate on simulating a Peano system within formal set theory, including a version of theorem 2 to permit recursive definitions. Once that is done, it can be maintained that formal set theory includes natural-number arithmetic. To accommodate study of what principles are really necessary for arithmetic, the formal set theory will adopt at first only those axioms absolutely required for this process. Later, additional and stronger axioms will permit nearly all common mathematical arguments to be recast in formal set theory, including those used to construct the real and complex number systems and to derive the theorems of algebra and analysis. Thus it will become apparent that nearly all contemporary mathematics can be carried out within formal set theory.

Dedekind took one of the first steps in this direction. In [1888] 1963, he defined a notion of infinite set and showed how to use set theory to construct a Peano system from any given infinite set.[3] The idea of an infinite set was controversial then. Many mathematicians regarded some sets as *potentially* infinite: new members can be invented, without regard to their number, as required to do mathematics. But Dedekind's construction of a Peano system $<N, ^+, 0>$ seemed to require that the elements of N be considered all at once, as the members of a *completed* set. Further, his method for incorporating arithmetic into set theory was criticized on grounds of economy of thought: why should it be necessary to wrestle with controversial ideas about *infinite* sets in order to justify simple manipulations with natural numbers, the cardinals of *finite* sets?

These notes use for formal set theory a first-order language with just one nonlogical symbol $\in$, intended to stand for membership. Its intended universe of discourse consists just of sets.[4] Its axioms should include at least the *axiom of extensionality*:

$$(\forall x, y)[\forall w[w \in x \iff w \in y] \Rightarrow x = y].$$[5]

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[3] Dedekind did suggest the existence of one infinite set, that of all objects of our thought. But this idea has generally been rejected as too vague for mathematics.

[4] It is easy to modify this formal theory to accommodate in the universe of discourse individuals, often called *Urelemente*, that are not sets but can be members of sets. This modification is not considered in these notes because it leads to clutter and is unnecessary for the task at hand: incorporating mathematics into set theory.

[5] $(\forall x, y)[x = y \Rightarrow \forall w[w \in x \iff w \in y]]$ is deducible from *logical* axioms. Some presentations of first-order logic, however, do not include a theory of equality: they require each application to define its own equality. If such a logic were in use here, $x = y$ would be defined as $\forall w[w \in x \iff w \in y]$.

As long as possible, these notes will avoid use of set-theoretic axioms that refers to any set that mathematicians would regard as infinite. The results derived in this formal theory without such axioms should be acceptable to those who believe that all sets are finite. (Or if they still object, that will have to be on other grounds!) The technique used here is due to Willard v. O. Quine in 1969.

The notion of Peano system itself only *apparently* involves an infinite set N, an operation $^+ : N \to N$, and a specified object $0 \in N$. In the formal set theory their roles can be played by formulas. The first of these will be written "n is natural"; it's introduced in more detail later. I won't completely specify the other two, called Z and Σ, until later, but from now on I'll assume that the following are theorems of the formal set theory:

$$\exists n \forall m [\mathrm{Z}(m) \Leftrightarrow m = n] \qquad \text{—Z describes a } constant$$

$$(\forall m,n,p)[\Sigma(m,n) \ \& \ \Sigma(m,p) \Rightarrow n = p] \qquad \text{—}\Sigma \text{ describes a } function$$

$$(\exists m,n)[\mathrm{Z}(m) \ \& \ \Sigma(m,n)] \ \& \ \forall m [\exists n \Sigma(m,n) \Rightarrow \exists p \Sigma(n,p)] \qquad \text{—}\Sigma \text{ is } iterative.$$

Henceforth, I'll employ virtual class techniques and write "$m = 0$" for "$\mathrm{Z}(m)$" and "$0 \in S$" for "$\exists m [m \in S \ \& \ \mathrm{Z}(m)]$". Similarly, I'll write "$m^+ = n$" for "$\Sigma(m,n)$". You must take care *not* to interpret $^+$ as an operator in the sense of first-order logic, because I'm not assuming that it's applicable to *every* object m. With these conventions, the iterative assumption can be rendered as

$$\exists n \Sigma(0,n) \ \& \ \forall m [\exists n [m^+ = n] \Rightarrow \exists p [m^{++} = p]].$$

Thus, the iterative assumption permits me to manipulate $0^+, 0^{++}, 0^{+++}, \ldots.$

Inspection of the statements and proofs of theorems 1 and 2 reveals that virtual-class techniques suffice there except in two steps noted in the paragraph after theorem 2. There I used variables S' and e' to indicate sets $S \cup \{n^+\}$ and $e \cup \{<m^+, g(m,e(m))>\}$ constructed by abstraction, and I used universal and existential generalization with those variables. Set-theoretic axioms are required there to guarantee the existence of *sets* S' and e'. Without that, S' and e' are merely virtual classes, not subject to manipulation with quantifiers. Each of these requires justification of the formation of the union of two given sets, and of the set consisting solely of one given member. Besides the apparent occurrences of such sets in those clauses, those processes also underlie the construction of ordered pairs. Therefore the formal set theory must include *at least these comprehension axioms*:

$$(\forall x,y)\exists z \forall w [w \in z \Leftrightarrow w \in x \vee w \in y] \qquad \text{—existence of } z = x \cup y$$

$$\forall x \exists z \forall w [w \in z \Leftrightarrow w = x] \qquad \text{—existence of } z = \{x\}.$$

Theorem 3 could also be reformulated to permit a proof using just these axioms. Instead of working with two given Peano systems, the result would be phrased in terms of two groups of formulas that describe the systems. In place of functions $g : N \times S \to$

S and $f: N \to S$, the proof would construct formulas that describe f and g. The isomorphism would not be a function in the set-theoretic sense, but a formula that would determine one if the sets N, S, and $N \times S$ existed.

Theorem 4 presents a problem. Although the definition given here for the $\le$ relation in a Peano system $<N, ^+, 0>$ involves quantification only over closed subsets $S \subseteq N$, which mathematicians would regard as finite, the equivalent description of $\le$ that appears in the theorem and the more common definition quantifies over infinite sets $T = N - S$. Mathematicians who believe only in finite sets could avoid explicit mention of the set N, but would regard the condition $(\forall T \subseteq N)\big[m \in T \ \& \ (\forall l \in T)[l^+ \in T] \Rightarrow n \in T\big]$ as true for *every* pair m,n because for them there are no such T;[6] they couldn't use this formula as a definition of $m \le n$. That is why these notes presented the alternative form of the definition.

The usual definition of "n is natural," due to Dedekind [1888] 1963, is

$$\forall S\big[0 \in S \ \& \ (\forall m \in S)[m^+ \in S] \Rightarrow n \in S\big].$$

In other words, the natural numbers should be those objects that *must* belong to any set that contains $0^+, 0^{++}, 0^{+++}, \ldots$. But that doesn't work for mathematicians who believe only in finite sets. No finite set S could satisfy the condition $(\forall m \in S)[m^+ \in S]$. According to that definition they would have to call *everything* natural. Instead, these notes use a modified definition suggested by these notes' definition of the relation $\le$ in a Peano system:

$$\begin{aligned} S \text{ is } \Sigma\text{-closed} &\Leftrightarrow \forall m[(\exists n \in S)\Sigma(m,n) \Rightarrow m \in S] \\ n \text{ is natural} &\Leftrightarrow (\exists \Sigma\text{-closed } S)[n \in S] \\ &\qquad \& \ (\forall \Sigma\text{-closed } S)\big[n \in S \Rightarrow 0 \in S\big]. \end{aligned}$$

The latter definition is due to Quine 1969, section 11, except that he omitted its first clause. In 1987 Charles D. Parsons recognized that as an error and suggested this correction, citing George 1987. The corrected definition works even for those who believe only in finite sets, because they'll regard closed sets as finite.

The following discussion requires two more assumptions about formulas Z and Σ:

$$(\forall m,n)[Z(n) \Rightarrow \neg\Sigma(m,n)] \qquad \text{—\textit{zero} assumption}$$
$$(\forall m,n,p)[\Sigma(m,p) \ \& \ \Sigma(n,p) \Rightarrow m = n] \qquad \text{—\textit{injectivity} assumption.}$$

Theorem 5

a. $\{0\}$ is Σ-closed.
b. 0 is natural
c. $(\forall \text{natural } m)\exists n\,\Sigma(m,n)$

[6] The infinite set N presents no problem because that variable isn't in the scope of any quantifier; phrases such as $n \in N$ can be replaced by formulas of formal set theory that do not mention N.

Proof. Part (a) stems from the zero assumption and immediately entails part (b). To prove part (c), suppose there were a natural m for which $\neg\exists n\,\Sigma(m,n)$. By the definition of naturalness, m would belong to some closed set S. Consider the set $Q = \{\, l \in S : \neg\exists n\,\Sigma(l,n)\,\}$, which would contain m. (The paragraph immediately after this proof justifiies this step.) By the iterative assumption and the fact that S is closed, so would be Q. By the definition of naturalness, $0 \in Q$, contrary to the iterative assumption. ♦

This proof used one of the set-theoretic comprehension principles called *separation axioms*. Therefore, these must now be included in the formal set theory: for each formula Φ in which the variable z is not free,

$$\forall S\exists z\forall l\,[\, l \in z \;\Leftrightarrow\; l \in S \;\;\&\;\; \Phi\,] \qquad \text{—existence of } z = \{\, l \in S : \Phi\,\}.$$

According to theorem 5c, the function $^+$ represented by Σ has a value for every natural argument. By the following theorem, that value is also natural.

Theorem 6. $(\forall \text{natural } m)[\, m^+ \text{ is natural}\,]$.

Proof. Suppose m is natural, so that it belongs to some Σ-closed set S. Then m^+ belongs to the set $T = S \cup \{m^+\}$. Moreover, T is Σ-closed: if $l^+ \in T$ for some l, then $l^+ \in S$, in which case $l \in S$ because S is closed, or else $l^+ = m^+$, in which case $l = m \in S$ by the injectivity assumption.

It remains to show that if T is any Σ-closed set containing m^+, then $0 \in T$. That is a consequence of the naturalness of m, because T would then be a Σ-closed set containing m. ♦

The next theorem shows that the natural numbers, as described here, satisfy the postulates for a Peano system $<N, {}^+, 0>$, except that they don't necessarily constitute a set N and $^+$ isn't necessarily a set. In part (c), the recursion postulate, ordinarily stated as

$$(\forall S \subseteq N)\,[\, 0 \in S \;\;\&\;\; (\forall n \in S)[\, n^+ \in S\,] \;\Rightarrow\; S = N\,]\,],$$

is reformulated as a scheme for a family of propositions, to avoid quantifying over infinite sets S and explicitly mentioning N. These propositions are stated separately for each formula Φ that might determine such an $S = \{\text{natural } m : \Phi\}$. Later, when

Theorem 7 (Dedekind–Peano postulates)

a. $(\forall \text{ natural } n)[\, 0 \neq n^+]$ —*zero*

b. $(\forall \text{ natural } m,n)[\, m^+ = n^+ \;\Rightarrow\; m = n\,]$ —*injectivity*

c. For every formula Φ,
$\Phi(0) \;\;\&\;\; (\forall \text{natural } m)[\,\Phi(m) \Rightarrow \Phi(m^+)\,]$
$\Rightarrow\; (\forall \text{natural } m)\,\Phi(m)$ —*recursion*

Proof. Parts (a, b) stem from the zero and injectivity assumptions. To prove part (c), suppose Φ is a formula such that $(\forall \text{natural } m)[\,\Phi(m) \Rightarrow \Phi(m^+)\,]$. Suppose m is natural, but $\neg\Phi(m)$. It suffices to prove that $\neg\Phi(0)$. Since m is natural, it belongs to a Σ-closed set S. Let $T = \{\, l \in S : \neg\Phi(l) \,\}$, so that $m \in T$ and T is Σ-closed. Since m is natural, $0 \in T$, as required. ♦

Earlier in these notes the notion of *closed* set played a central role in the discussion of Peano systems. To avoid confusion, the next theorem shows that it is essentially the same as the notion of Σ-closed set; the proof is straightforward.

Theorem 8. A set of natural numbers is closed if and only if it is Σ-closed.

As noted earlier, you can follow one of the standard texts to define the binary addition, multiplication, and exponentiation operators in the Peano system just described, and to prove their properties, which mimic the familiar arithmetic of natural numbers. You should find the interface between these notes and that presentation smooth, except for the need to avoid explicitly mentioning infinite sets. For example, the binary addition operator should be defined as follows. First, let $\mathrm{A}(m,p,q)$ stand for the formula

$$\begin{aligned}&m \text{ is natural } \ \&\ p \text{ is natural}\\ &\quad \&\ (\exists \text{ function } f)\,[\,(\forall \text{ natural } n \le p)(\exists \text{natural } y)[\,y = f(n)\,]\\ &\qquad \&\ f(0) = m \ \&\ (\forall n < p)[\,f(n^+) = f(n)^+\,]\,].\end{aligned}$$

Next, prove the following theorems by recursion on p:

$$(\forall \text{ natural } m,p)(\exists \text{ natural } q)\,\mathrm{A}(m,p,q)$$

$$(\forall \text{ natural } m,p,q,r)[\,\mathrm{A}(m,p,q) \ \&\ \mathrm{A}(m,p,r) \Rightarrow q = r\,].$$

Thus, A can represent a binary operator on natural numbers m,p: we can write $m + p = q$ to stand for $\mathrm{A}(m,p,q)$.

In 1969 Quine, following Zermelo [1908] 1970a, defined the formulas Z and Σ as follows, and carried out in detail the development of natural-number arithmetic:

$\mathrm{Z}(m)$: $\quad m = \phi$

$\Sigma(m,n)$: $\quad n = \{\, m \,\}$ —*Zermelo's successor formula*

It is particularly easy to verify

Theorem 9. The constant, function, iterative, zero, and injectivity assumptions for Zermelo's formulas Z and Σ are theorems of formal set theory. Thus, according to his definition of naturalness, formal set theory incorporates natural-number arithmetic.

With Zermelo's definitions, the natural numbers are $\phi, \{\phi\}, \{\{\phi\}\}, \ldots$. This definition is probably the simplest. Essentially it builds into formal set theory an apparatus that amounts to assessing the syntactical complexity of the formulas that determine the natural numbers. It seems to me that the only criticism of Zermelo's method is that it doesn't apply to much else in mathematics.

Another method, due to John von Neumann and others, is more common today.[7] It is almost as simple, and enjoys the advantange that it can be used for the foundation of ordinal- and cardinal-number theory as well. The added complexity is its use of an auxiliary definition: a set m is called *transitive* if $(\forall l,m)[l \in m \in n \Rightarrow l \in n]$. Here are von Neumann's formulas Z and Σ.

$$\begin{aligned} &\mathrm{Z}(m): && m = \phi \\ &\Sigma(m,n): && m \text{ is transitive} \\ & && \quad \& \ n = m \cup \{m\} \qquad \text{—}\textit{von Neumann's successor formula} \end{aligned}$$

Verifying the following theorem is a routine exercise.

Theorem 10. The constant, function, iterative, zero, and injectivity assumptions for the von Neumann formulas Z and Σ are theorems of formal set theory. Thus, according to that definition of naturalness, formal set theory incorporates natural-number arithmetic.

With the von Neumann definitions, the natural numbers are $0 = \phi$, $1 = \{0\}$, $2 = \{0,1\}$, $3 = \{0,1,2\}$, ... In fact, each natural number n is the set of those preceding it, a set whose cardinal number is n! These definitions are adopted now for use later in this course.

Applying the *entire* theory of Peano systems in formal set theory requires adopting more set-theoretic axioms, because Peano systems are defined as sets and studied using everyday mathematics. For example, an *axiom of infinity* is required. Its form usually depends on the formulas Z and Σ employed for defining "n is natural":

$$(\exists S,z)[\mathrm{Z}(z) \ \& \ z \in S \ \& \ (\forall m \in S)(\exists n \in S)\Sigma(m,n)].$$

This and a separation axiom entail

$$\begin{aligned} &\exists \mathbb{N} \forall x [x \in \mathbb{N} \\ &\quad \Leftrightarrow x \in S \ \& \ \forall Q[z \in Q \ \& \ (\forall m \in Q)(\exists n \in Q)\Sigma(m,n)] \Rightarrow x \in Q], \end{aligned}$$

and that leads to a definition and theorem:

$$\mathbb{N} = \bigcap\{Q : z \in Q \ \& \ (\forall m \in Q)(\exists n \in Q)\Sigma(m,n)\} = \{x : x \text{ is natural}\}.$$ [8]

Mathematicians who would develop a comprehensive natural-number arithmetic under weaker set-theoretic assumptions must reconstruct under those assumptions the required parts of the theory of Peano systems.

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[7] See Quine 1969, section 22, footnote 2.

[8] The axiom of infinity can in fact be assumed for any one of the possible choices of Z and Σ, if the formal set theory incorporates the *axiom of replacement*. Use theorem 2 to construct a formula defining an isomorphism from the Peano system based on that choice of Z and Σ to the virtual class that represents the desired system. By the axiom of replacement, the range of that isomorphism is a set.

Routine exercises

1. a. Find an algebraic structure $<N, {}^+, 0>$, where ${}^+$ is an operator on N and $0 \in N$, that satisfies the injectivity and recursion postulate but not the zero postulate.

   b. Find one that satisifies the zero and recursion postulate but not the injectivity postulate.

   c. Find one that satisfies the zero and injectivity postulates but not the recursion proof postulate.

2. a. Under the hypotheses for theorem 7 prove that for every formula Φ for which $\forall n\,[\Phi(n) \Rightarrow n$ is natural$]$ is a theorem of set theory, so is

   $$\exists n\,\Phi(n) \Rightarrow \exists m)\,[\Phi(m) \;\&\; \forall l\,[\Phi(l) \Rightarrow \neg\Sigma(l,m)]].$$

   This result can often be used in place of recursion to clean up messy proofs.

   b. Rephrase and reprove part (a) in the context of a Peano system.[9]

   c. Let $<N, {}^+, 0>$ be a Peano system, S be any set, $z \in S$, and $g : N \times \mathscr{P}S \to S$. Prove that there exists a unique $f : N \to S$ such that

   $$f(0) = z \;\&\; (\forall n \in N)\,[f(n) = g(n, \{f(m) : m < n\})].$$

   This result justifies an alternative form of recursive definition.

   d. Why does the discussion after theorem 2 *not* show how to rephrase part (c) in the context of formulas Z and Σ?

3. Prove theorem 10.

Substantial problem

1. A set is called *hereditarily finite* if its members are finite, as are members of its members, members of their members, and so on. Using set theory beyond that covered in this course that notion can be made precise, and a set H can be constructed that consists of all the hereditarily finite sets. Of course H itself is infinite. Let $\in_H$ stand for the binary relation $\{<x,y> \in H \times H : x \in y\}$.

   a. Which axioms of set theory are valid in the structure $<H, \in_H>$ and which not?

   b. Assume that the natural numbers $\mathbb{N}$ and the integers $\mathbb{Z}$ are disjoint subsets of H. Let 0 denote the zero in $\mathbb{N}$, and let ${}^+$ denote the union of the opera-

[9] In [1907] 2007, Mario Pieri showed that the result of exercise 2b can be used in place of the recursion postulate in the definition of a Peano system, and that doing so makes the injectivity postulate derivable from the others, hence dispensible.

tors $n \rightsquigarrow n+1$ on $\mathbb{N}$ and $\mathbb{Z}$. Which of the Dedekind–Peano postulates does the structure $<\mathbb{N} \cup \mathbb{Z}, {}^+, 0>$ satisfy, and which not?

c. Now let $\Sigma(m,n)$ be the formula $m^+ = n$, where $^+$ is construed as in part (b). Which sets are Σ-closed? How does this show that Quine's original definition of naturalness was inadequate?