MAXIMAL PRINCIPLES

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Bourbaki's Fixpoint Theorem

Let $<X, \le>$ be a nonempty partially ordered set. A nonempty subset $C \subseteq X$ is called a *chain* if

$$(\forall x, y \in C)[\, x \le y \;\vee\; y \le x \,].$$

Assume that each chain $C \subseteq X$ *has a supremum* (in X, but not necessarily in C). *Suppose* $f : X \to X$ *and* $x \le f(x)$ *for all* $x \in X$. *It follows that* $f(x) = x$ *for some* $x \in X$. Nicolas Bourbaki announced this result in 1939. The remainder of this section of these notes is a roadmap of the proof. It's not necessary to use the axiom of choice.

Let $a \in X$. Throughout the proof a remains fixed. A subset $A \subseteq X$ is called *admissible* if

1) $a \in A$
2) $x \in A \Rightarrow f(x) \in A$
3) if F is a chain and $F \subseteq A$, then $\bigvee F \in A$.

Clearly, X is itself an admissible subset. Let $\mathcal{A}$ denote the family of all admissible subsets of x, and define

$$M = \bigcap \mathcal{A}.$$

It's easy to check that M is admissible. *Thus* M *is the smallest admissible subset*. Now define

$$M' = \{x \in M : a \le x\}.$$

It's easy to check that M', too, is admissible, hence $M \subseteq M'$ and therefore $M = M'$. *Thus* a *is the minimum element of* M.

For each $x \in M$, let

$$B_x = \{y \in M : y \le x \;\vee\; f(x) \le y\}.$$

It's tedious, but not really difficult, to check that B_x is admissible if $(\forall y \in M)[\, y < x \Rightarrow f(y) \le x \,]$. In that case, again, $M \subseteq B_x$, hence $M = B_x$. This argument has just shown that

$$\begin{aligned} &x \in M \;\&\; (\forall y \in M)[\, y < x \Rightarrow f(y) \le x \,] \\ &\quad \Rightarrow (\forall y \in M)[\, y \le x \;\vee\; f(x) \le y \,]. \end{aligned} \tag{*}$$

Now define

$$C = \{x \in M : (\forall y \in M)[\, y < x \;\Rightarrow\; f(y) \le x\,]\}.$$

Again, it's tedious, but not difficult, to check that C is admissible, hence $M \subseteq C$ and therefore $M = C$. You can thus simplify formula (*):

$$x \in M \;\Rightarrow\; (\forall y \in M)[\, y \le x \;\vee\; f(x) \le y\,].$$

But $x \le f(x)$ for all x, hence

$$x \in M \;\Rightarrow\; (\forall y \in M)[\, y \le x \;\vee\; x \le y\,].$$

That is, *M is a chain*.

Because M is admissible, $M \subseteq M$, and M is a chain, it follows that $\bigvee M \in M$. Again because M is admissible, $f(\bigvee M) \in M$, hence $f(\bigvee M) \le \bigvee M$. But $\bigvee M \le f(\bigvee M)$ also, hence $f(\bigvee M) = \bigvee M$. This completes the proof of Bourbaki's fixpoint theorem.

Originally, mathematicians used the axiom of choice most frequently to show that some set X could be well-ordered—that there existed a linear ordering relation on X with respect to which each nonempty subset of X had a minimum element. Then they'd use the involved theory of recursion in a well-ordered set to define some function on X. However, Felix Hausdorff, Casimir Kuratowski, and Max Zorn noticed independently in 1914, 1922, and 1935 that most of these tedious detours via the theory of well-ordered sets could be replaced by simple references to a maximal principle which is in fact an alternative form of the axiom of choice. As a result the theory of well-ordered sets has little use in mathematics today outside pure set theory. The tedious mathematics of recursive definitions in well-ordered sets has been transformed into Bourbaki's fixpoint theorem. In the next section of these notes, the maximal principle will appear as an easy corollary of that result and the axiom of choice.

Maximal Principles

Let $\langle X, \le \rangle$ be a partially ordered set and $A \subseteq X$ be a chain. Then A is included in some "maximal" chain $B \subseteq X$ —that is, $A \subseteq B$ and B is included in a chain $C \subseteq X$ only if $B = C$. This is Hausdorff's 1914 version of the maximal principle.

The following *proof* is due to Helmut Kneser in 1950. Suppose the conclusion were false. Let

$$\mathcal{K} = \{K \subseteq X : A \subseteq K \;\&\; K \text{ is a chain}\}.$$

For each $K \in \mathcal{K}$ set

$$\mathcal{K}_K = \{L \in \mathcal{K} : K \subseteq L \;\&\; K \ne L\}.$$

Then $\mathcal{K}_K \ne \phi$ for each $K \in \mathcal{K}$, so by the axiom of choice there exists a choice function

$$f \in \bigtimes_{K \in \mathcal{K}} \mathcal{K}_K.$$

Then $f : \mathcal{K} \to \mathcal{K}$ and for each $K \in \mathcal{K}$, $K \subseteq f(K)$ but $K \neq f(K)$. Consider the partially ordered set $<\mathcal{K}, \subseteq>$. If $\mathcal{F} \subseteq \mathcal{K}$ is a chain—that is, $(\forall K, L \in \mathcal{F})[K \subseteq L \vee L \subseteq K]$ —then $\bigcup\mathcal{F} \in \mathcal{K}$. Therefore, by Bourbaki's fixpoint theorem, $f(K) = K$ for some $K \in \mathcal{K}$, contradiction!

The axiom of choice is unnecessary in the preceding proof when a choice function c *for* X *is known*, for then you can define $f(K) = K \cup \{c(\{x \in X : K \cup \{x\} \in \mathcal{K}_K\})$. This is the case, for example, when X is the range of a sequence—that is, $\mathrm{X} = \{x_n : n \in \mathbb{N}\}$.

It's only now possible to derive another, seemingly elementary, finiteness property: *a set* U *is infinite if and only if* $\omega \leq \#U$. The *if* part of this equivalence was derived in the notes *Cardinals I*. To verify the *only if* part, let U be infinite and consider the partially ordered set $<F, \subseteq>$, where F is the family of all f such that for some $n \in \mathbb{N}$, $f : \{m \in \mathbb{N} : m < n\} \to U$ injectively. By the maximal principle, F has a maximal chain B, so that $\bigcup B$ is an injection from some subset of $\mathbb{N}$ to U. Clearly, $0 \in \operatorname{Dom} \bigcup B$. Suppose, for sake of argument, that $\operatorname{Dom} \bigcup B \neq \mathbb{N}$. Consider the minimum $n \in \mathbb{N} - \operatorname{Dom} \bigcup B$, so that $n > 0$, $n - 1 \in \operatorname{Dom} \bigcup B$, and hence $n - 1 \in \operatorname{Dom} f$ for some $f \in B$. If $m \in \mathbb{N}$ and $m < n - 1$ then $m \in \operatorname{Dom} f \subseteq \operatorname{Dom} \bigcup B$; thus, $\operatorname{Dom} \bigcup B = \{m \in \mathbb{N} : m < n\}$, and hence $\bigcup B \in F$. Since U is infinite, $\{m \in \mathbb{N} : m < n\} \not\approx U$, so $\operatorname{Rng} \bigcup B \neq U$, and there would exist $u \in U - \operatorname{Rng} \bigcup B$; let $g = (\bigcup B) \cup \{<n, u>\}$. Then $g \in F$ and $\bigcup B \subsetneq g$, but $g \notin B$ because $u \in \operatorname{Rng} g$. Therefore, $B \subsetneq B \cup \{g\}$, a chain in F, which would contradict the maximality of B. Thus, the assumption $\operatorname{Dom} \bigcup B \neq \mathbb{N}$ was false, so $\mathbb{N} \preceq U$, which yields $\omega \leq \#U$ as desired.

A set U is called *Dedekind infinite* just when it has an equinumerous proper subset P. *No finite set is Dedekind infinite*, because then $\#U = \#P + \#(U - P)$ and $U - P$ is nonempty. The result of the previous paragraph, implies that *every infinite set* U *is Dedekind infinite*, for in that case there is an injection $f : \mathbb{N} \to U$, the function $e : U \to U$ defined by setting

$$e(x) = \begin{cases} f(f^{-1}(x) + 1) & \text{if } x \in \operatorname{Rng} f \\ x & \text{otherwise} \end{cases}$$

for all $x \in U$ is injective, and its range does not include $f(0)$. Thus the axiom of choice implies that the notions of infinite and Dedekind infinite sets are equivalent.

In 1922 Kuratowski independently proved the following alternative form of the maximal principle. *Let* $<X, \leq>$ *be a partially ordered set, each chain* C *of which has an upper bound* (*in* X, *not necessarily in* C). *Then* $<X, \leq>$ *has a maximal element*. To prove this, apply Hausdorff's maximal principle to find a maximal chain $M \subseteq X$; let m be an upper bound for M. For each $x \in X$, $m \leq x \Rightarrow M \cup \{x\}$ is a chain $\Rightarrow M \cup \{x\} = M \Rightarrow x \in M \Rightarrow x \leq m \Rightarrow m = x$. Thus m is a maximal element of X.

In proving the Kuratowski maximal principle also, the axiom of choice is unnecessary when a choice function c *for* X *is known: in particular, if* X *is the range of a sequence.*

Suppose $\{A_i : i \in I\}$ is an indexed family of nonempty sets. Define

$$X = \bigcup_{J \subseteq I} \bigtimes_{i \in J} A_i .$$

Then $<X, \subseteq>$ is a partially ordered set. Moreover, if $K \subseteq X$ is a chain, then $\bigcup K \in K$. By the Kuratowski maximal principle, X has a maximal element c. Now,

$$c \in \bigtimes_{i \in J} A_i .$$

for some $J \subseteq I$. Therefore, $J = I$ since c is maximal. *This argument shows that the Kuratowski maximal principle, hence Hausdorff's, is equivalent to the axiom of choice.*

Zermelo's [1904] 1970 proof that every set can be well-ordered was similar to the proofs of Bourbaki's fixpoint theorem and Hausdorff's maximal principle given here. Mathematicians raised many objections, which Zermelo answered in [1908] 1970b, as follows. (1) Zermelo used the theory of well-ordered sets, in which Cesare Burali-Forti had found a contradiction in [1897] 1970. Zermelo showed that his result used only uncontested parts of the suspect theory. (2) The axiom of choice had not been proved. Zermelo pointed out that neither had other basic principles, and that many accepted results depended on the axiom. (3) The axiom of choice provided *no effective method* of choice, and Zermelo's argument *no construction* of a well-ordering even for the set of all real numbers. This objection led to a systematic effort to distinguish constructive from nonconstructive arguments, and to provide constructive arguments whenever possible. (4) The last objection, partially formulated in the [1905] 1982 series of letters by Émile Borel, Henri Lebesgue, René Baire, and Henri Poincaré, pertained to the entire theory of infinite sets. An infinite set should not be regarded as a completed entity—only as a virtual collection whose members come into being from time to time as they're constructed or defined in the process of mathematical creation. For example, the treatment of $\mathcal{A}$ and M on Page 1 is objectionable because M, whose definition requires complete knowledge of $\mathcal{A}$, must itself be a member of $\mathcal{A}$. This situation is called *impredicativity*. In response to these objections, several alternative systems of foundations have developed: predicative, intuitionist, and constructivist theories. All these deny large parts of modern mathematics.

Routine exercises

1. A family $\mathcal{F}$ of subsets of a set X is said to have *finite character* if for each $S \subseteq X$, $S \in \mathcal{F}$ if and only if every finite subset of S belongs to $\mathcal{F}$. Prove that every such $\mathcal{F}$ has a maximal member. This result is due to Oswald Teichmüller and John Tukey, independently, in 1939 and 1940.

2. A partial ordering $\leq$ of a set X is called a *well-ordering* if each nonempty subset of X has a minimum element. Show that every well-ordering is linear. Prove that the following proposition is equivalent to the axiom of choice: Every nonempty set has a well-ordering. This is due to Ernst Zermelo in [1904] 1970.

3. Prove Edward Szpilrajn's 1930 result that each partial ordering of a set X is included in some linear ordering of X.

4. Let $\mathcal{F}$ be a family of subsets of a set X. Show that $\mathcal{F}$ has a maximal subfamily $\mathcal{M}$ such that $(\forall K,L \in \mathcal{M})[K \neq L \Rightarrow K \cap L = \phi]$.

5. Let $\mathcal{F}$ be a family of subsets of a set X. Show that $\mathcal{F}$ has a maximal subfamily $\mathcal{M}$ such that $(\forall K,L \in \mathcal{M})[K \cap L \neq \phi]$.

6. Let $<X, \leq>$ be a partially ordered set in which every chain has a supremum. Show that every homomorphism from X to X has a fixpoint.

Substantial problems

1. Prove that the result of routine exercise 1 implies the axiom of choice.

2. Prove that a partial ordering of a set X is included in some well-ordering of X if and only if every nonempty subset of X has a minimal element.

3. Consider a family $\{<X_i, \leq_i> : i \in I\}$ of partially ordered sets. Let $P = \times_i X_i$ and define a relation $\leq$ on P by setting, for each $f,g \in P$,

   $$f \leq g \Leftrightarrow \forall i\,[\,f_i \leq_i g_i\,].$$

   a. Show that $\leq$ is a partial ordering of P. The partially ordered set $<P, \leq>$ is called the *product* of the $<X_i, \leq_i>$.

   b. Show that every partially ordered set $<X, \leq>$ is isomorphic to a subset of a product of linearly ordered sets. Hint: Take all the $X_i = X$.

4. Show that the results of routine exercises 4 and 5 are each equivalent to the axiom of choice. These equivalences are due to Robert L. Vaught in 1952 and Djuro Kurepa in 1952. Hints:

   a. Let $\mathcal{X}$ be a family of nonempty disjoint sets and consider

   $$\{\{<0,v>,<1,W>\} : v \in W \in \mathcal{X}\}.$$

   b. Let $\mathcal{F}$ be a family of sets and for each $S \in \mathcal{F}$ consider

   $$\mathcal{N}_S = \{S\} \cup \{\{S,T\} : T \in \mathcal{F} \;\&\; S \cap T = \phi\}.$$

5. Derive from Bourbaki's fixpoint theorem the recursion theorem for well-ordered sets: Given a well-ordered set $<X, \le>$, a set Y, and a function

   $$g : X \times \mathcal{P}(X \times Y) \to Y,$$

   there's a unique function $f : X \to Y$ such that for each $x \in X$,

   $$f(x) = g(<x, \{<w, f(w)> : w \in X \;\&\; w < x\}>). \qquad (*)$$

   Hint: Consider the family of all functions f defined on "initial segments" of X and satisfying (*).

6. A *filter* of a lattice $<L, \le>$ is a nonempty subset $F \subseteq L$ such that for all $x, y \in L$,

   $$x \in F \;\&\; y \in F \Leftrightarrow x \wedge y \in F.$$

   Clearly, L is itself a filter; all others are called *proper*.

   a. Show that the filters form a Moore family of subsets of L (and hence a complete lattice).

   b. Show that if $x \in F$ and F is a filter of L, then $x\!\uparrow\; \subseteq F$.

   c. Show that for any $x \in L$, $x\!\uparrow$ is a filter of L. Such a filter is called *principal*.

   d. Clearly, a one-element lattice has no proper filter. Show that every other lattice has at least one proper filter.

   A proper filter F of L is called an *ultrafilter* if the only filters that contain it are F and L.

   e. Find a lattice with no minimum element and no ultrafilter.

   A filter F is called *prime* if

   $$x \in F \;\vee\; y \in F \Leftrightarrow x \vee y \in F$$

   for all $x, y \in L$. Clearly, the improper filter is prime.

   f. Show that in a distributive lattice, every ultrafilter is prime. Hint: show that if $x, y \in L$ and F is a filter of L containing $x \vee y$, then $G = \{w \in L : (\exists z \in F)[y \wedge z \le w]\}$ is a filter of L that contains y and includes F.

g. Find a nondistributive lattice with five or fewer elements in which every ultrafilter is prime, and another in which some ultrafilter is not prime. Display Hasse diagrams for the lattices of all filters of these examples.

7. An *ideal* of a lattice $<L, \leq>$ is a nonempty subset $I \subseteq L$ such that

$$x \in I \ \& \ y \in I \Leftrightarrow x \vee y \in I$$

for all $x, y \in L$; it is called *prime* if also

$$x \in I \vee y \in I \Leftrightarrow x \wedge y \in I.$$

a. Formulate statements analogous to parts (a–f) of substantial problem 6, with ideals instead of filters. It's unnecessary to prove these, since the notion *ideal of L* coincides with *filter of the dual of L*.

The analog of the ultrafilter theorem says that *every lattice with a maximum element and at least one other has a maximal proper ideal*. This result is called the *maximal ideal theorem*. For the same reason, it's unnecessary to prove that.

b. Show that the maximal ideal theorem implies the Hausdorff maximal principle (and hence is equivalent to the axiom of choice). This result is due to Dana Scott in 1954. Hint: given a chain C of an arbitrary partially ordered set $<X, \leq>$, let $\mathcal{L} = \{\phi, X\} \cup \{K \subseteq X : C \subseteq K \ \& \ K \text{ is a chain}\}$, show that $<\mathcal{L}, \subseteq>$ is a Moore family of subsets of X, and consider its union.

c. Find a three-element partially ordered set X for which the lattice $\mathcal{L}$ in part (d) is not distributive.

8. A *lattice of sets* is a family $\mathcal{L}$ of sets that contains the union and intersection of any two of its members— $<\mathcal{L}, \subseteq>$ is thus a distributive lattice, and the supremum and infimum of any two of its members are their union and intersection. By substantial problem 7, the axiom of choice implies that *every lattice of sets with a minimum member and at least one other has an ultrafilter*. Show that this statement implies the axiom of choice. This result is due to John L. Bell and David H. Fremlin, in 1972. Hints:

a. Let r be a relation. The problem is to find a function $q \subseteq r$ such that $\operatorname{Dom} q = \operatorname{Dom} r$. Show first that the family F of all functions contained in r is nonempty, and consider the partially ordered set $<F, \supseteq>$.

b. For each $f \in F$, let $S(f) = \{g \in F : g \supseteq f\}$. Review the properties of the function S described in the *Partially Ordered Sets* notes and in *Complete Lattices* substantial problem 7.

c. Let $\mathcal{L} = \{\bigcup S[E] : E \text{ is finite } \& \ E \subseteq F\}$. Show that $\mathcal{L}$ is a lattice of sets, with minimum member ϕ and at least one other. The italicized statement implies that $\mathcal{L}$ has an ultrafilter $\mathcal{U}$. Consider $\bigcup S^{-1}[\mathcal{U}]$.

References

Heise 1971.
Kanamori 1997.
Kneser 1950.
Stoll [1963] 1979, 111–118.