# EQUIVALENCES AND PARTITIONS

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An *equivalence* on a set $X$ is a binary relation $E$ on $X$ that's *reflexive*, *symmetric*, and *transitive*: for all $x, y, z \in X$,

$$x \, E \, x \qquad\qquad x \, E \, y \;\Rightarrow\; y \, E \, x \qquad\qquad x \, E \, y \;\&\; y \, E \, z \;\Rightarrow\; x \, E \, z \, .$$

This is the mathematical explication of the informal concept *sameness*. Clearly, the identity $I_X$ is the smallest equivalence on $X$, and $X \times X$ is the largest.

Let $E$ be an equivalence on $X$. Each $x \in X$ belongs to its *equivalence class*

$$x/E = \{ y \in X : x \, E \, y \} \, .$$

Two equivalence classes $x/E$ and $y/E$ are equal just when $x \, E \, y$. The function $x \mapsto x/E$ is called the *quotient map*. Its range is the family $X/E$ of all equivalence classes.

Suppose $f : X \to Y$ surjectively. Define a binary relation $E_f$ on $X$ by setting

$$x \, E_f \, y \;\Leftrightarrow\; f(x) = f(y)$$

for all $x, y \in X$. Then $E_f$ is an equivalence on $X$. Moreover, you can construct a bijection $g : X/E_f \to Y$ such that for each $x \in X$, $g(x/E_f) = f(x)$. Proceed as follows. First, show that there's a unique $y \in Y$ —namely $y = f(x)$ —such that $f(x') = y$ for all $x' \in x/E_f$. Then define $g(x/E_f) = y$.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {}_{/E_f}\searrow & & \nearrow_{g} \\ & X/E_f & \end{array}$$

In the previous paragraph, if $Y = X/E$ for some equivalence $E$ on $X$, and $f : X \to Y$ is the quotient map, then $E_f = E$ and $g$ is the identity $I_Y$.

A *partition* of $X$ is a family of disjoint subsets of $X$ whose union is $X$. This is the mathematical explication of the informal concept *classification*. You've an example already: if $E$ is an equivalence on $X$, then $X/E$ is a partition of $X$. The partition corresponding to the equivalence $I_X$ consists of all singletons of members of $X$; it's the *finest* partition. The *coarsest* partition corresponds to the equivalence $X \times X$: it consists of the set $X$ alone.

Suppose $\mathcal{F}$ is a partition of $X$. Define a binary relation $E_{\mathcal{F}}$ on $X$ by setting

$$x \, E_{\mathcal{F}} \, y \;\Leftrightarrow\; \exists A \, [\, A \in \mathcal{F} \;\&\; x, y \in A \,]$$

for all $x, y \in X$. Then $E_{\mathcal{F}}$ is an equivalence on $X$, and $X/E_{\mathcal{F}} = \mathcal{F}$.

In the previous paragraph, if $\mathcal{F} = X/E$ for some equivalence $E$ on $X$, then $E_{\mathcal{F}} = E$.

These results show that the notions *equivalence* and *partition* are merely two aspects of the same concept.

## Routine Exercises

1. Show that a relation $E$ on $X$ is

   a) reflexive if and only if $I_X \subseteq E$;
   b) symmetric if and only if $E \subseteq \breve{E}$;
   c) transitive if and only if $E|E \subseteq E$.

   Show that for an equivalence $E$ on $X$, actually $E = \breve{E}$ and $E|E = E$.

2. Define binary relations $E$ and $F$ on the set $X$ of Americans by setting for each $x, y \in X$,

   $x\,E\,y$ if and only if $x$ and $y$ have the same age, in years;
   $x\,F\,y$ if and only if $x$ and $y$ differ in age by less than one year.

   Are $E$ and $F$ equivalences on $X$?

3. Find a relation on the set $\{0,1\}$ that's nonempty, symmetric, and transitive, but not reflexive.

4. Show that the intersection of a nonempty family of equivalences on $X$ is an equivalence on $X$. Let $\mathcal{E}$ be a nonempty family of equivalences on $X$ such that for any $D, E \in \mathcal{E}$ there exists $F \in \mathcal{E}$ with $D, E \subseteq F$. Show that $\bigcup\mathcal{E}$ is an equivalence on $X$.

5. Let $\mathcal{F}$ and $\mathcal{F}'$ be partitions of $X$. $\mathcal{F}$ is called *finer* than $\mathcal{F}'$ if each member of $\mathcal{F}$ is included in some member of $\mathcal{F}'$. Show that this is true just when $E \subseteq E'$, where $E$ and $E'$ are the equivalences corresponding to $\mathcal{F}$ and $\mathcal{F}'$.

## Substantial problems

1. Let $R$ be a relation on $X$. Define $R^1 = R$ and for each integer $n > 1$,

   $$R^n = R|\ldots|R \quad (n \text{ copies of } R).$$

   Define $R^* = \bigcup_n R^n$, the *transitive closure* of $R$. Prove that $R^*$ is the smallest transitive relation on $X$ that includes $R$.

2. Let $R$ be a relation such that $R|\breve{R}|R \subseteq R$ and set $X = \operatorname{Dom} R$. Prove that $R|\breve{R}$ is an equivalence on $X$.

3. Let $H(n)$ denote the number of partitions of an $n$-element set. Show that

$$H(n+1) = \sum_{m=0}^{n} \binom{n}{m} H(m) .$$

Compute $H(10)$. Prove that for all real $x$,

$$e^{e^x - 1} = \sum_{n=0}^{\infty} \frac{H(n)}{n!} x^n.$$

(See Birkhoff 1948, 108, exercise 1.)

4. Let $E$ and $F$ be equivalences on $X$ and $E|F = F|E$. Prove that $E|F$ is the smallest equivalence on $X$ that includes $E$ and $F$.

5. For each integer $n > 0$, define a binary relation $E_n$ on $\mathbb{Z}$ by setting, for all integers $a$ and $b$, $a\, E_n\, b$ if and only if $n$ divides $a - b$. Show that $E_n$ is an equivalence on $\mathbb{Z}$. Now consider two integers $m$ and $n > 0$. Show that $E_m \subseteq E_n$ if and only if $n$ divides $m$. Let $p$ and $q$ be the least common multiple and greatest common divisor of $m$ and $n$. Show that $E_m \cap E_n = E_p$ and $E_m|E_n = E_n|E_m = E_q$.

## Reference

Kurosh 1963, 18–20.