

COMPLETE LATTICES

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Consider a partially ordered set $\langle X, \leq \rangle$ and a subset $Y \subseteq X$. An element x of X is called an *upper bound* of Y if $y \leq x$ for every $y \in Y$; it's a *lower bound* of Y if $x \leq y$ for every $y \in Y$. If the set of upper bounds of Y has a minimum element, it's called the *supremum* of Y , written $\bigvee Y$. If the set of lower bounds of Y has a maximum element, it's called the *infimum* of Y , written $\bigwedge Y$. Clearly, if Y has a maximum element x , then $x = \bigvee Y$. Moreover, if Y has a supremum x , and $x \in Y$, then x is the maximum element of Y . Taking $Y = X$, you can see that the notions "supremum of X " and "maximum of X " coincide. Similar statements hold for infimum and minimum. Taking $Y = \phi$, you can see that the notions "supremum of ϕ " and "minimum of X " coincide, as do "infimum of ϕ " and "maximum of X ."

If every subset of X has a supremum and an infimum, then $\langle X, \leq \rangle$ is called a *complete lattice*. For example, consider $\langle \mathcal{P}S, \subseteq \rangle$ for any set S , $\langle I, \leq \rangle$ for any bounded interval I of natural numbers, and $\langle I, \leq \rangle$ for any closed bounded interval I of real numbers. The partially ordered set $\langle \mathbb{Z}, \leq \rangle$ is not a complete lattice because \mathbb{Z} has no supremum. For the same reason, $\langle I, \leq \rangle$ is not a complete lattice if I is the interval $[0, 2)$ of real numbers. Consider the interval $I = \{x \in \mathbb{Q} : 0 \leq x \leq 2\}$ of rational numbers: $\langle I, \leq \rangle$ is not a complete lattice because $\{x \in I : x^2 < 2\}$ has no supremum.

The definition of "complete lattice" can be simplified: you need only check that every subset of X has an infimum, for the supremum of a set is then the infimum of the set of all its upper bounds. Dually, you could just verify that every subset of X has a supremum.

A nonempty family \mathcal{F} of sets is called a *Moore family* if it has a maximum member X and contains the intersection of each of its nonempty subfamilies:

$$S \in \mathcal{F} \Rightarrow S \subseteq X \quad \phi \neq \mathcal{S} \subseteq \mathcal{F} \Rightarrow \bigcap \mathcal{S} \in \mathcal{F}.$$

$\langle \mathcal{F}, \subseteq \rangle$ is then a complete lattice because each of its subfamilies has an infimum. For nonempty subsets $\mathcal{S} \subseteq \mathcal{F}$, we have $\bigwedge \mathcal{S} = \bigcap \mathcal{S}$, while $\bigwedge \phi = X$. Moreover, for any $\mathcal{S} \subseteq \mathcal{F}$,

$$\bigvee \mathcal{S} = \bigcap \{T : \bigcup \mathcal{S} \subseteq T\}.$$

There are many familiar Moore families in algebra, including the subgroups of a group, its normal subgroups, the subrings of a ring, and its ideals.

A *closure operation* on a nonempty set X is a function $c : \mathcal{P}X \rightarrow \mathcal{P}X$ such that for all $A, B \subseteq X$,

$$A \subseteq c(A) \qquad c(c(A)) = c(A) \qquad A \subseteq B \Rightarrow c(A) \subseteq c(B).$$

From these properties of c you can easily derive the following rules: if $A_i \subseteq X$ for all $i \in I$, then

$$\bigcup_i c(A_i) \subseteq c(\bigcup_i A_i) \qquad c(\bigcap_i A_i) \subseteq \bigcap_i c(A_i).$$

A subset $A \subseteq X$ is called *closed* under c if $c(A) = A$. Let \mathcal{F}_c denote the family of all closed subsets of X . Apparently, \mathcal{F}_c is the range of c . Moreover, X is itself closed and the intersection of any nonempty family of closed sets is closed. Thus \mathcal{F}_c is a Moore family.

Now let \mathcal{F} be any Moore family of subsets of X and define $c_{\mathcal{F}} : \mathcal{P}X \rightarrow \mathcal{P}X$ by setting

$$c_{\mathcal{F}}(A) = \bigcap \{B \in \mathcal{F} : A \subseteq B\}$$

for every $A \subseteq X$. You can check easily that for all $A \subseteq X$, $c_{\mathcal{F}}(A) = A \Leftrightarrow A \in \mathcal{F}$, and that $c_{\mathcal{F}}$ is a closure operation on X .

If in the previous paragraph \mathcal{F} is the family of all subsets of X that are closed under some closure operation c , then $c = c_{\mathcal{F}}$. Moreover, if c is a closure operation on X that's induced, as just described, by a Moore family \mathcal{F} , then $\mathcal{F}_c = \mathcal{F}$. Thus the notions "closure operation" and "Moore family" are just two aspects of the same concept.

The following *fixpoint theorem*—due to Alfred Tarski in 1955—is used in the theory of cardinals: if $\langle X, \leq \rangle$ is a complete lattice, $f : X \rightarrow X$, and

$$x \leq y \Rightarrow f(x) \leq f(y)$$

for all $x, y \in X$, then $f(x_0) = x_0$ for some $x_0 \in X$.

Proof. Let

$$x_0 = \bigvee \{x \in X : x \leq f(x)\}.$$

Then for all $x \in X$, $x \leq f(x) \Rightarrow x \leq x_0 \Rightarrow f(x) \leq f(x_0) \Rightarrow x \leq f(x_0)$. Therefore $x_0 \leq f(x_0)$; this implies $f(x_0) \leq f(f(x_0))$, and thus $f(x_0) \leq x_0$.

Routine Exercises

1. A partially ordered set $\langle X, \leq \rangle$ is called *conditionally complete* if (i) each subset of X that has an upper bound has a supremum, and (ii) each subset of X that has a lower bound has an infimum. Show that (i) holds if and only if (ii) does, so that this definition could be simplified. Suppose $\langle X, \leq \rangle$ is conditionally complete. Let α and β be two objects not in X and define

$$-\infty = \begin{cases} \bigwedge X & \text{if this exists} \\ \alpha & \text{otherwise} \end{cases} \quad +\infty = \begin{cases} \bigvee X & \text{if this exists} \\ \beta & \text{otherwise} \end{cases}$$

$$X^* = X \cup \{-\infty, +\infty\}$$

$$\leq^* = \{ \langle -\infty, x \rangle : x \in X^* \} \cup \leq \cup \{ \langle x, +\infty \rangle : x \in X^* \}.$$

Show that $\langle X^*, \leq^* \rangle$ is a complete lattice, and that it's a linearly ordered set if $\langle X, \leq \rangle$ is.

2. Let V be a two-inch by four-inch rectangular region, and define

$$\begin{aligned} \mathcal{P} &= \{ \{p\} : p \text{ is a point in } V \} \\ \mathcal{K} &= \{ K : K \text{ is a circular region in } V \} \\ \mathcal{X} &= \mathcal{P} \cup \mathcal{K}. \end{aligned}$$

Then $\langle \mathcal{X}, \subseteq \rangle$ is a partially ordered set. Describe any maximum, minimum, maximal, or minimal members of \mathcal{X} . What suprema and infima exist? Is $\langle \mathcal{X}, \subseteq \rangle$ a complete lattice? How does the situation change if you include the empty set in \mathcal{X} ? The region V itself?

3. A partially ordered set $\langle X, \leq \rangle$ in which each two-element subset has a supremum and an infimum is called a *lattice*. For each $x, y \in X$, define

$$x \vee y = \bigvee \{x, y\} \quad x \wedge y = \bigwedge \{x, y\}.$$

Prove the following rules:

$x \vee x = x$	<i>Idempotency</i>
$x \wedge x = x$	
$x \vee y = y \vee x$	<i>Commutativity</i>
$x \wedge y = y \wedge x$	
$x \vee (y \vee z) = (x \vee y) \vee z$	<i>Associativity</i>
$x \wedge (y \wedge z) = (x \wedge y) \wedge z$	
$x \vee (x \wedge y) = x$	<i>Absorptivity</i>
$x \wedge (x \vee y) = x$	
$x \vee y = y \Leftrightarrow x \leq y \Leftrightarrow x \wedge y = x.$	

Show that every linearly ordered set is a lattice. Display Hasse diagrams for all lattices with five or fewer elements. Show that all finite lattices are complete.

4. Display Hasse diagrams for the families of all subsets of sets of one, two, three, and four elements; and for the families of all equivalences on sets of one, two, three, and four elements.
5. Let $\langle X, \leq \rangle$ be a complete lattice, I be a nonempty set, and $x : I \times I \rightarrow X$. Show that

$$\bigvee_j \bigwedge_i x_{ij} \leq \bigwedge_i \bigvee_j x_{ij}.$$

Find examples to show that \leq can be $=$ or \neq .

6. Let $\langle X, \leq \rangle$ be a complete lattice. For each $a, b \in X$, define the *closed interval*

$$[a, b] = \{x \in X : a \leq x \text{ \& } x \leq b\}.$$

Show that the intersection of any nonempty family of closed intervals is a closed interval.

7. Let $\langle X, \leq \rangle$ be a partially ordered set and define $\varphi : X \rightarrow \mathcal{P}X$ by setting, for each $x \in X$,

$$\varphi(x) = \{t \in X : t \leq x\}.$$

My notes “Partially ordered sets” showed that φ is an injective homomorphism from $\langle X, \leq \rangle$ to $\langle \mathcal{P}X, \subseteq \rangle$. Suppose $y_0 \in X$, $Y \subseteq X$, and $y_0 = \bigwedge Y$. Show that

$$\bigcap_{y \in Y} \varphi(y) = \varphi(y_0).$$

You can say then that φ “preserves infima.” Give an example to show that φ doesn’t necessarily “preserve suprema.”

8. In Tarski’s fixpoint theorem, consider $x = \bigwedge \{x \in X : x \geq f(x)\}$.

Substantial problems

1. A lattice $\langle X, \leq \rangle$ is called *modular* if for all $x, y, z \in X$,

$$x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z.$$

Show that the normal subgroups of a group form a complete modular lattice. Show that there’s just one isomorphism class of lattices with five or fewer elements that aren’t modular.

2. A lattice $\langle X, \leq \rangle$ is called *distributive* if

- i) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all $x, y, z \in X$
 ii) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x, y, z \in X$.

Show that (i) holds if and only if (ii) does, so that the definition could be simplified. Show that every distributive lattice is modular. Show that there are just two isomorphism classes of lattices with five or fewer elements that aren’t distributive.

3. Consider the complete lattice $\langle \mathcal{E}, \subseteq \rangle$ where \mathcal{E} is the family of all equivalences on some nonempty set X . Let $\mathcal{A} \subseteq \mathcal{E}$. Show that for all $x, y \in X$, $x (\bigvee \mathcal{A}) y$ if and only if there exist $E_1, \dots, E_n \in \mathcal{A}$ such that $x (E_1 \mid \dots \mid E_n) y$. Show that when X has more than three elements, this lattice is not modular.

References

Birkhoff 1948 is the bible of this area of mathematics. See also the third edition, 1967, which differs considerably.

Kurosh 1963, 22–26.