CARDINALS II

James T. Smith
San Francisco State University

These notes develop the part of cardinal arithmetic that depends on the axiom of choice.

The first result is the *comparability theorem: every nonempty set of cardinals has a minimum element*. The following *proof* was given by Chaim S. Hönig in 1954. Consider an arbitrary family $\{A_i : i \in I\}$ of sets. The problem is to show that $\exists i\, \forall j\, [A_i \preceq A_j]$. That's obvious if A_i is empty for some i, so assume the opposite. By the axiom of choice, there exists $a \in \mathsf{X}_i A_i$. Define

$$\mathscr{X} = \{\, X \subseteq \mathsf{X}_i A_i : (\forall x, y \in X)\,[\, x \neq y \;\Rightarrow\; \forall i\, [\, x_i \neq y_i\,]\,]\,\}.$$

Then $\mathscr{X} \neq \phi$ because $\{a\} \in \mathscr{X}$. Consider the partially ordered set $<\mathscr{X}, \subseteq>$. You can easily check that if $\mathscr{C} \subseteq \mathscr{X}$ is a chain, then $\bigcup \mathscr{C} \in \mathscr{X}$. By Kuratowski's maximal principle, $\mathscr{X}$ has a maximal element M. For each $i \in I$, define

$$M_i = \{x_i : x \in M\} \subseteq A_i .$$

If $\forall i\, [A_i - M_i \neq \phi\,]$, then there would exist $b \in \mathsf{X}_i (A_i - M_i)$ by the axiom of choice; you could verify that $M \cup \{b\} \in \mathscr{X}$, which would imply $b \in M$ because M is maximal, and that would yield the contradiction $\forall i\, [\, b_i \in M_i\,]$. Therefore, $\exists i\, [M_i = A_i\,]$. If $t \in A_i$, then $t = x_i$ for some $x \in M$, and this x is unique: for if $x, y \in M$ and $x_i = y_i = t$, then $x = y$ because $M \in \mathscr{X}$. Thus for each j, you can define a function $f_j : A_i \to A_j$ by setting $f_j(t) = x_j$. Each f_j is injective: if $f_j(t) = f_j(u)$ for some $t, u \in A_i$, then there would exist $x, y \in M$ for which

$$\begin{array}{ll} t = x_i & \qquad f_j(t) = x_j \\ u = y_i & \qquad f_j(u) = y_j , \end{array}$$

and hence $x_j = y_j$, which implies $x = y$ because $M \in \mathscr{X}$. Therefore, $A_i \preceq A_j$ for all j. The proof is complete.

It follows from the comparability theorem that $\leq$ is a linear ordering on any set of cardinals. That provides another proof *that a set* X *is infinite if and only if* $\omega \leq \#X$. Sets of cardinality $\leq \omega$ are called *countable*.

The first step in the development of the arithmetic of infinite cardinals is to show that $\omega + \omega = \omega$. You can see that by considering the bijection $f : \mathbb{N} \to (\mathbb{N} \times \{0\}) \cup (\mathbb{N} \times \{1\})$ described by the following diagram:

$$\begin{array}{ccccccc} <0,0> & & <1,0> & & <2,0> & & \ldots \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \\ <0,1> & & <1,1> & & <2,1> & & \ldots \end{array}$$

As a corollary, you can deduce that $n + \omega = \omega$ for each $n \in \mathbb{N}$.

The next step is the equation $\alpha + \omega = \alpha$ for each infinite cardinal α. To prove that, let $\alpha = \#X$, C be a countably infinite subset of X, and D be a countably infinite set disjoint from X. There's a bijection $f: C \cup D \to C$. Use it to define a bijection $g: X \cup D \to X$ by setting

$$g(x) = \begin{cases} f(x) & \text{if } x \in C \cup D \\ x & \text{if } x \in X - C \end{cases}.$$

That completes the proof. As a corollary, you can deduce that $n + \alpha = \alpha$ for each $n \in \mathbb{N}$ and each infinite cardinal α.

The next result is the equation $\alpha + \alpha = \alpha$ first proved for *every* infinite cardinal α by Gerhard Hessenberg in 1906. To prove it, let $\alpha = \#X$ and

$$F = \{f : (\exists S \subseteq X)[f : (S \times \{0\}) \cup (S \times \{1\}) \to S \text{ injectively}]\}.$$

By considering any countably infinite $S \subseteq X$, you can see that F is nonempty. Now consider the partially ordered set $\langle F, \subseteq \rangle$. You can easily check that if $C \subseteq F$ is a chain, then $\bigcup C \in F$. By Kuratowski's maximal principle, therefore, F has a maximal member f. Let $\operatorname{Dom} f = (S \times \{0\}) \cup (S \times \{1\})$ and $\beta = \#S$, so that $\beta + \beta \le \beta$. The desired equation follows if $S \approx X$. Suppose, on the contrary, that $S \prec X$. Then $X - S$ would be infinite, for otherwise $\alpha = \beta + n = \beta$, where $n = \#(X - S)$, and $X - S$ would have a countably infinite subset T. There would exist an injection $g: (T \times \{0\}) \cup (T \times \{1\}) \to T$, and $f \cup g$ would be an injection from $((S \cup T) \times \{0\}) \cup ((S \cup T) \times \{1\})$ to $S \cup T$. Since f is maximal, $f \cup g = f$, contradiction! The proof is complete.

Hessenberg's sum theorem implies that *the sum of two infinite cardinals is the larger of them*: if $\omega \le \alpha \le \beta$, then $\beta \le \alpha + \beta \le \beta + \beta = \beta$. By a recursive argument, it follows that *the sum of any finite set of infinite cardinals is their maximum.*

You can generalize the argument given earlier for the equation $\omega + \omega = \omega$ —which is equivalent to $2\omega = \omega$ —to yield $n\omega = \omega$ for each $n \in \mathbb{N}$. In fact, $\omega\omega = \omega$. You can see that by considering the bijection $f: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ described by the following diagram:

$$\begin{array}{cccccccc}
\langle 0,0\rangle & & \langle 1,0\rangle & \to & \langle 2,0\rangle & & \langle 3,0\rangle & \to \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & \iddots \\
\langle 0,1\rangle & & \langle 1,1\rangle & & \langle 2,1\rangle & & \langle 3,1\rangle & \\
& \swarrow & & \nearrow & & \swarrow & & \iddots \\
\langle 0,2\rangle & & \langle 1,2\rangle & & \langle 2,2\rangle & & \langle 3,2\rangle & \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & \iddots \\
\langle 0,3\rangle & & \langle 1,3\rangle & & \langle 2,3\rangle & & \langle 3,3\rangle & \\
& \iddots & & \iddots & & \iddots & &
\end{array}$$

Now you can prove that $\alpha\alpha = \alpha$ for *every* infinite cardinal α. Hessenberg also proved this first in 1906. The following proof, and the one presented earlier for the equation $\alpha + \alpha = \alpha$, are patterned after a 1944 argument of Max Zorn. Let $\alpha = \#X$ and

$$F = \{f : (\exists S \subseteq X)[f : S \times S \to S \text{ injectively}]\}.$$

By considering any countably infinite $S \subseteq X$, you can see that F is nonempty. Now consider the partially ordered set $\langle F, \subseteq \rangle$. You can easily check that if $C \subseteq F$ is a chain, then $\bigcup C \in F$. By Kuratowski's maximal principle, therefore, F has a maximal member f. Let $\operatorname{Dom} f = S \times S$ and $\beta = \#S$, so that $\beta\beta \le \alpha$. The desired equation follows if $S \approx X$. Suppose, on the contrary, that $S \prec X$. Then $\#(X - S) \ge \beta$, for otherwise $\alpha = \beta + \gamma = \beta$, where $\gamma = \#(X - S)$. There would be a subset $T \subseteq X - S$ such that $\#T = \beta$, and

$$\#[(S \times T) \cup (T \times T) \cup (T \times S)] = \beta\beta + \beta\beta + \beta\beta = \beta + \beta + \beta = \beta.$$

There would exist an injection g from $(S \times T) \cup (T \times T) \cup (T \times S)$ to T, and $f \cup g$ would be an injection from $(S \cup T) \times (S \cup T)$ to $S \cup T$. Since f is maximal, $f \cup g = f$, contradiction! The proof is complete.

Hessenberg's product theorem implies that *the product of two infinite cardinals is the larger of them*: if $\omega \le \alpha \le \beta$, then $\beta \le \alpha\beta \le \beta\beta = \beta$. By a recursive argument, it follows that *the product of any finite set of infinite cardinals is their maximum*.

The set $\mathbb{Z}$ of all integers has cardinal $\omega + \omega = \omega$, since $\mathbb{Z} = \mathbb{N} \cup \{-n : n \in \mathbb{N}\}$. Let X denote the set of all pairs $\langle m,n \rangle \in \mathbb{Z} \times \mathbb{N}$ such that $n \ne 0$ and m and n are relatively prime. Clearly, $X \approx \mathbb{Q}$, the set of all rational numbers, hence $\#\mathbb{Q} \le \#(\mathbb{Z} \times \mathbb{N}) = \omega\omega = \omega$. *Thus* $\mathbb{Z}$ *and* $\mathbb{Q}$ *are countably infinite.*

Let X denote the set of all sequences of zeros and ones that aren't eventually all ones:

$$X = \{x \in \{0,1\}^{\mathbb{N}} : \neg(\exists m)(\forall n \ge m)[x_n = 1\}.$$

By considering binary expansions, you can see that $X \approx [0,1)$, the half-open unit interval of real numbers. Therefore $\#[0,1) \le 2^{\omega}$. But there's an injection $f : \{0,1\}^{\mathbb{N}} \to X$ defined by setting, for each $x \in \{0,1\}^{\mathbb{N}}$,

$$f(x)_n = \begin{cases} x_{n/2} & \text{if } x \text{ is even} \\ 0 & \text{if } x \text{ is odd.} \end{cases}$$

Thus the half-open unit interval has cardinal 2^{ω}.

Let a and b be real numbers and $a < b$. Consider the bijection $f : [0,1) \to [a,b)$ defined by the equation $f(t) = (1 - t)a + tb$: apparently, *every* half-open interval has cardinal 2^{ω}. From the diagram below, you can see that every *bounded* interval of real numbers has cardinal 2^{ω}.

[[([( ]))])

Binary expansion of real numbers indicates that $\mathbb{Z} \times [0,1) \approx \mathbb{R}$, the set of real numbers. Thus $\#\mathbb{R} = \omega 2^\omega = 2^\omega$. *Therefore all intervals of real numbers, bounded or unbounded, have cardinal* 2^ω. That cardinal is called the *continuum*. This result was proved by Georg Cantor in 1878.

The question whether there's any cardinal β such that $\omega < \beta < 2^\omega$ has been found undecidable within mathematicians' usual set theory. The proposition that *no* such β should exist is called the *continuum hypothesis*. That is equivalent to asserting that *every* uncountable set of real numbers should have cardinal 2^ω. Certain special cases *can* be derived within this set theory. For instance, Cantor and Ivar Bendixson showed in 1883 that every uncountable closed set of real numbers has cardinal 2^ω, and Paul Alexandroff, Felix Hausdorff, and M. Souslin showed the same in 1916 for every uncountable measurable set. In spite of this evidence, many mathematicians doubt the validity of the continuum hypothesis.

Routine exercises

1. Prove that for any cardinals n and α,
   a. $0 < n < \omega \Rightarrow \omega^n = \omega$
   b. $0 < n \le \omega \Rightarrow (2^\omega)^n = 2^\omega$
   c. $2 \le \alpha \le 2^\omega \Rightarrow \alpha^\omega = 2^\omega$
   d. $2 \le \alpha \le 2^\omega \Rightarrow \alpha^{2^\omega} = 2^{2^\omega}$.

2. Show that the cardinal of the set of all irrational numbers is the continuum.

3. Let I be a nonempty countable set and suppose A_i is a countable set for each $i \in I$. Prove that $\bigcup_i A_i$ is countable. Point out where you use the axiom of choice in your proof.

4. Let $f : \mathbb{R} \to \omega$. Show that there exists a subset $S \subseteq \mathbb{R}$ whose cardinality is the continuum, such that $f(x) = f(y)$ for all $x, y \in S$. Hint: use the Zermelo-König inequality.

5. Show that $\#\{s \in \mathbb{N}^\mathbb{N} : \forall n\,[s_{n+1} \le s_n]\} = \omega$.

6. Show that $\#\{s \in \mathbb{N}^\mathbb{N} : \forall n\,[s_n \le s_{n+1}]\} = 2^\omega$.

7. Show that $\mathbb{N}^{\mathbb{N}}$ isn't countable by supposing it were and changing the nth term of the nth sequence, for each $n \in \mathbb{N}$. Apply a similar argument to the half-open interval $[0,1)$. Cantor introduced this elegant *diagonal* argument in [1891] 1966, though he had proved that $[0,1)$ was uncountable already in 1874, by a much more complicated method.

8. Find a bijection f from $\mathbb{R}$ to the open interval $(0,1)$ that's familiar to students of calculus.

9. Prove that $\mathbb{R} \times \mathbb{R} \approx \mathbb{R}$ by "intertwining" decimal expansions. Cantor originally proved this result in 1878 by a similar method, using continued fractions instead of decimal expansions.

10. Find a function $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that for every function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ there exists $g : \mathbb{R} \to \mathbb{R}$ for which $f = g \circ h$.

11. Without using the axiom of choice show that the proposition that $X \preceq Y$ or $\mathscr{P}Y \preceq X$ for all infinite sets X and Y implies that the order relation between cardinals is linear, and implies the *generalized continuum hypothesis*, that for any infinite cardinals α, β, the inequalities $\alpha \leq \beta \leq 2^\alpha$ imply $\alpha = \beta$ or $\beta = 2^\alpha$. This 1966 result is due to Rolf Schock.

12. Show that the cardinal of the set of all convergent sequences of real numbers is the continuum.

Substantial problems

1. Let $\mathscr{L}$ denote the set of all lines in the plane, and

$$\mathscr{L}_0 = \{L \in \mathscr{L} : \#(L \cap \mathbb{Q}^2) = 0\}$$
$$\mathscr{L}_1 = \{L \in \mathscr{L} : \#(L \cap \mathbb{Q}^2) = 1\}$$
$$\mathscr{L}_2 = \{L \in \mathscr{L} : \#(L \cap \mathbb{Q}^2) > 1\}.$$

Prove

a. $L \in \mathscr{L} \;\Rightarrow\; \#L = 2^\omega$
b. $L \in \mathscr{L}_2 \;\Rightarrow\; \#(L \cap \mathbb{Q}^2) = \omega$
c. $\#\mathscr{L}_0 = 2^\omega$
d. $\#\mathscr{L}_1 = 2^\omega$
e. $\#\mathscr{L}_2 = \omega$.

2. Let X be an infinite set and suppose α is a nonzero cardinal $\le \#X$. Show that X is the union of an indexed family of disjoint sets $\{A_x : x \in X\}$ such that $\#A_x = \alpha$ for each $x \in X$.
3. Let X be an infinite set and $\beta = \#X$.
   a. Show that there exists a bijection $\pi : X \to X$ such that $\pi(x) \neq x$ for each $x \in X$. (Use $\alpha = \omega$ in substantial problem 2.)
   b. Show that $\#\{A \subseteq X : \#A = \beta\} = 2^\beta$.

4. Let X be an infinite set and $\beta = \#X$.
   a. Show that $\beta^\beta = 2^\beta$.
   b. Show that $\#\{A \subseteq X : \#(X - A) > 1\} = 2^\beta$.
   c. Show that the set of all bijections from X to X has cardinality 2^β. (Use substantial problem 3a.) This could be expressed as β *infinite* $\Rightarrow \beta! = 2^\beta$.
   d. What about the sets of all injections and of all surjections?

5. Let X be an infinite set and $\beta = \#X$.
   a. Show that the set of all equivalence relations on X has cardinal 2^β.
   b. Show that the set of all linear orderings of X has cardinal 2^β. (Use substantial problem 4c.)
   c. What about the set of all partial orderings?

6. Let X be an infinite set and $\beta = \#X$. Suppose α is a cardinal such that $2^\alpha \le \alpha^\alpha \le \beta$.
   a. Show that $2^\alpha \le \beta^\alpha \le 2^\beta = \alpha^\beta$.
   b. Show that $\#\{A \subseteq X : \#A = \alpha\} = \beta^\alpha$.
   c. Let Y be a set with cardinal α. Show that the set of all injections from Y to X has cardinal β^α.

7. A cardinal δ is called *dominant* if $\alpha^\beta < \delta$ for all cardinals $\alpha, \beta < \delta$. For example, ω is the smallest dominant cardinal.
   a. Show that a cardinal δ is dominant if and only if $2^\beta < \delta$ for each cardinal $\beta < \delta$.
   b. Find a sequence $\{A_n : n \in \mathbb{N}\}$ of disjoint sets such that $\#A_0 = \omega$ and for all $n \in \mathbb{N}$, $\#A_{n+1} = 2^{\#A_n}$.
   c. Show that $\delta = \#\bigcup_n A_n$ is a dominant cardinal $> \omega$.
   d. Show that no cardinal γ such that $\omega < \gamma < \delta$ is dominant.
   e. Show that $2^\delta = \delta^\omega$.
   f. Show that $\delta^\omega = \omega^\delta$.

8. a. Show that for any cardinals $\alpha, \beta, \gamma, \delta$,

$$\alpha < \gamma \ \& \ \beta < \delta \Rightarrow \alpha + \beta < \gamma + \delta \ \& \ \alpha\beta < \gamma\delta.$$

b. Find nonzero cardinals $\alpha, \beta, \gamma, \delta$ such that $\alpha < \gamma$, $\beta < \delta$, and $\alpha^{\beta} = \gamma^{\delta}$. (Use substantial problem 7e.)

9. a. Show that the cardinal of the family of all open subsets of $\mathbb{R}$ is the continuum.
   b. Same for closed subsets.

10. a. Show that the cardinal of the set of all continuous real-valued functions on the closed unit interval U is the continuum.
    b. Same for the set of all limits of pointwise convergent sequences of continuous functions on U.

11. a. The *Borel* sets are the members of the smallest family of subsets of $\mathbb{R}$ that contains the open sets and is closed under the operations of complementation and formation of the union of a sequence of sets. Show that the cardinal of the family of all Borel sets is the continuum.
    b. The *Baire* functions are the members of the smallest family of real-valued functions on the closed unit interval U that contains the continuous functions and is closed under the operation of taking the limit of a pointwise convergent sequence of functions. Show that the cardinal of the family of all Baire functions is the continuum.

12. a. Show that the cardinal of the family of all measurable subsets of $\mathbb{R}$ is the continuum.
    b. Show that the cardinal of the set of all monotonic functions from $\mathbb{R}$ to $\mathbb{R}$ is the continuum.
    c. Show that the cardinal of the set of all Riemann integrable functions on the closed unit interval is $2^{2^{\omega}}$.
    d. Show that the cardinal of the set of all measurable functions from $\mathbb{R}$ to $\mathbb{R}$ is $2^{2^{\omega}}$.

13. Let $n \in \mathbb{N}$. Compute the following cardinals:
    a. $\#\{S \subseteq \mathbb{N} : \#S = n\}$
    b. $\#\{S \subseteq \mathbb{R} : \#S = n\}$
    c. $\#\{S \subseteq \mathbb{R} : \#S = \omega\}$.

14. a. For quasi-ordered sets $<X_1, \le_1>$ and $<X_2, \le_2>$, write $<X_1, \le_1> \, \le \, <X_2, \le_2>$ to indicate that there exists an isomorphism from $<X_1, \le_1>$ into (but not necessarily onto) $<X_2, \le_2>$. Show that $\le$ is a quasi-ordering relation on any family of quasi-ordered sets.

    b. Let I be a nonempty set and suppose that for each $i \in I$, $<A_i, \le_i>$ is a well-ordered set. Show that $\exists i \, \forall j \, [\, <A_i, \le_i> \, \le \, <A_j, \le_j> \,]$. Don't use the axiom of choice in your proof. (Hint: mimic Hönig's proof of the comparability theorem for cardinals. Consider the family

$$\mathscr{X} = \{ X \subseteq \textstyle\times_i A_i : (\forall x, y \in X) [\exists i\, (x_i \leq_i y_i) \Rightarrow \forall i\, (x_i \leq_i y_i)] \}.$$

$<\mathscr{X}, \subseteq>$ is a partially ordered set in which each chain has a supremum. Show that $(\exists X \in \mathscr{X})(\exists i)(\forall a \in A_i)(\exists x \in X)[a \leq_i x_i]$ by assuming its negation, defining a strictly increasing function from $\mathscr{X}$ to $\mathscr{X}$, and applying Bourbaki's fixpoint theorem to get a contradiction.)

c. Let $<X, \leq>$ be a quasi-ordered set, and construct a partially ordered set $<X/E, \leq/E>$ in the usual way by setting

$$x\,E\,y \Leftrightarrow x \leq y \;\&\; y \leq x \qquad x/E\,(\leq/E)\,y/E \Leftrightarrow x \leq y.$$

Suppose that for each nonempty $S \subseteq X$ there exists $x \in S$ such that $x \leq y$ for all $y \in S$. Show that $\leq/E$ is a well-ordering.

15. a. Let $<X, \leq>$ be a well-ordered set and f be an isomorphism from $<X, \leq>$ into (but not necessarily onto) itself. Show that $x \leq f(x)$ for each $x \in X$.

b. Let $<X, \leq>$ be a well-ordered set, $x \in X$, and $W = \{x : w < x\}$. Show that $<X, \leq> \not\preceq <W, \leq>$.

c. Let M be a nonempty set and Φ be the aggregate of all families of subsets of M that are well-ordered by the inclusion relation. Define a quasi-ordering relation $\leq$ on Φ as in substantial problem 14a, an equivalence relation E on Φ and a well-ordering $\leq/E$ of Φ/E as in substantial problem 14c. Show that there's no injection from Φ/E to M. Don't use the axiom of choice in your proof. (Hint: suppose $\Phi/E \preceq M$; find $L \subseteq M$ equinumerous with Φ/E; find a well-ordering R of L making $<\Phi/E, \leq/E>$ isomorphic to $<L, R>$; use Birkhoff's representation theorem to find $\mathscr{L} \in \Phi$ so that $<L, R>$ is isomorphic to $<\mathscr{L}, \subseteq>$; now find an isomorphism from $<\Phi/E, \leq/E>$ to

$$<\{ \mathscr{K}/E : \mathscr{K}/E\,(\leq/E)\,\mathscr{L}/E \;\&\; \mathscr{K}/E \neq \mathscr{L}/E \}, \leq/E>$$

—contradiction!)

d. An *aleph* is a cardinal of a set that has a well-ordering. Without using the axiom of choice, prove Wacław Sierpinski's 1947 theorem (stated but not proved in Lindenbaum and Tarski 1924, 314) that for each cardinal α there exists an aleph $\aleph$ such that

$$\aleph \leq 2^{2^{2^{\alpha}}} \;\&\; \aleph \not\leq \alpha.$$

e. Demonstrate F. Hartogs's 1915 result: the axiom of choice is implied by the proposition that for all cardinals α and β, $\alpha \leq \beta$ or $\beta \leq \alpha$.

f. Show that the axiom of choice is implied by the proposition that for all infinite cardinals α,β,γ,

$$\alpha < \gamma \ \ \& \ \ \beta < \gamma \ \Rightarrow \ \alpha + \beta \neq \gamma.$$

This result is attributed to Stanisław Lesniewski in the 1920s (see the MR review of Iseki 1950).

From (e) it's easy to derive the equivalence of the axiom of choice and either of

$$\alpha,\beta \text{ infinite } \ \& \ \alpha \le \beta \ \Rightarrow \ \alpha + \beta = \beta$$
$$\alpha,\beta \text{ infinite } \ \& \ \alpha \le \beta \ \Rightarrow \ \alpha\beta = \beta.$$

Alfred Tarski showed in 1924 that the equation $\alpha^2 = \alpha$ for infinite α is also equivalent. Gershon Sageev proved in 1973 that the corresponding law $2\alpha = \alpha$ for infinite α is weaker than the axiom of choice.

References

Cantor [1895–1897] 1952.

Fraenkel 1961: the most readable informal treatment of cardinal and ordinal arithmetic.

Gödel 1947.

Monk 1969. This axiomatic treatment of Morse's version of set theory presents cardinal arithmetic via the theory of ordinals. It contains a particularly good presentation of the recursion theorems.

Rubin and Rubin 1985.

Rubin 1967: comparable to Monk 1969, but a bit more detailed.

Sierpinski 1958: comparable to Fraenkel 1961, but much more detailed.

Suppes 1960. This axiomatic treatment of Zermelo–Fraenkel set theory presents cardinal arithmetic independently, not via the theory of ordinals.