CARDINALS I

James T. Smith
San Francisco State University

If there exists an injection from a set X to a set Y, then X is said to be *at most as numerous as* Y, written $X \preceq Y$. For all sets X, Y, and Z,

$$X \preceq X \qquad \text{—}\textit{reflexivity} \tag{1}$$
$$X \preceq Y \ \&\ Y \preceq Z \Rightarrow X \preceq Z \qquad \text{—}\textit{transitivity}. \tag{2}$$

Thus, $\preceq$ is a quasi-ordering relation on any family of sets. The argument that underlies reflexivity also yields

$$X \subseteq Y \Rightarrow X \preceq Y \tag{3}$$
$$\phi \preceq Y. \tag{4}$$

If $X \preceq Y$ & $Y \preceq X$, then X and Y are said to be *equinumerous*, written $X \approx Y$. For all sets X, Y, and Z,

$$X \approx X \qquad \text{—}\textit{reflexivity} \tag{5}$$
$$X \approx Y \Leftrightarrow Y \approx X \qquad \text{—}\textit{symmetry} \tag{6}$$
$$X \approx Y \ \&\ Y \approx Z \Rightarrow X \approx Z \qquad \text{—}\textit{transitivity}. \tag{7}$$

So far, this discussion parallels the construction of a partially ordered epimorph of a given quasi-ordered set: $\approx$ is an equivalence on any family $\mathcal{F}$ of sets, and $\preceq$ induces a partial ordering on $\mathcal{F}/\approx$.

The following characterization of the $\approx$ relation was conjectured by G. Cantor, and proved by Felix Bernstein in 1897: *for all sets X and Y, $X \approx Y$ if and only if there exists a bijection from X to Y. Proof.* If there exists a bijection from X to Y, then $X \approx Y$ trivially. Conversely, suppose $X \approx Y$. Then there exist injections $g : X \to Y$ and $h : Y \to X$. Define $F : \mathcal{P}X \to \mathcal{P}X$ by setting, for each $A \subseteq X$,

$$F(A) = X - h[Y - g[A]].$$

It follows that

$$A \subseteq B \Rightarrow F(A) \subseteq F(B).$$

By Tarski's fixpoint theorem applied to F and the complete lattice $<\mathcal{P}X, \subseteq>$, there exists $A \subseteq X$ such that

$$F(A) = A.$$

Further,

$$A = X - h[Y - g[A]] \qquad X - A = h[Y - g[A]] \qquad \breve{h}[X - A] = Y - g[A].$$

Define $f: X \to Y$ by setting

$$f(x) = \begin{cases} g(x) & \text{if } x \in A \\ \breve{h}(x) & \text{if } x \in X - A \end{cases}.$$

The restriction of f to A is a bijection from A to $g[A]$, and its restriction to $X - A$ is a bijection from $X - A$ to $Y - g[A]$. It follows that $f: X \to Y$ bijectively.

To classify sets according to the $\approx$ relation it's tempting to exploit the previously mentioned parallel with the theory of quasi-ordered sets. The members of any given family $\mathcal{F}$ are indeed partitioned into disjoint classes of equinumerous sets. Unfortunately, we can't handle *all* sets this way simultaneously, for there's no "universal" family $\mathcal{F}$. (In fact, $\mathcal{F}$ can't even contain all singletons, for then its union would be a universal set.)

This difficulty is bypassed by accepting as a basic notion of set theory an operation that assigns to each set X an object $\#X$ called its *cardinal*, so that for all sets X, Y,

$$X \approx Y \Leftrightarrow \#X = \#Y.$$

You can prove recursively that for any $m, n \in \mathbb{N}$,

$$\{k \in \mathbb{N} : k < m\} \preceq \{k \in \mathbb{N} : k < n\} \Leftrightarrow m \le n \tag{8}$$

$$\{k \in \mathbb{N} : k < m\} \approx \{k \in \mathbb{N} : k < n\} \Leftrightarrow m = n. \tag{9}$$

Therefore, for notational convenience, it's postulated that

$$\#\{k \in \mathbb{N} : k < m\} = m.$$

Moreover, we define $\omega = \#\mathbb{N}$.

A set X is called *finite* if $\#X \in \mathbb{N}$ —that is, if there's a bijection from X to $\{k \in N : k < m\}$ for some $m \in \mathbb{N}$. For finite sets X and Y, therefore, $X \preceq Y$ if and only if $\#X \le \#Y$. Moreover, for any sets X, Y, X' and Y',

$$X \approx X' \ \& \ Y \approx Y' \Rightarrow [\, X \preceq Y \Leftrightarrow X' \preceq Y' \,]. \tag{10}$$

These results permit the definition of an order relation $\le$ between cardinals:

$$\#X \le \#Y \Leftrightarrow X \preceq Y.$$

It follows that for all sets X and Y,

$$X \subseteq Y \Rightarrow \#X \le \#Y, \tag{11}$$

and that for all cardinals α, β, and γ,

$$0 \le \alpha \tag{12}$$

$$\alpha \le \alpha \qquad \text{—\textit{reflexivity}} \tag{13}$$

$$\alpha \le \beta \ \& \ \beta \le \alpha \Rightarrow \alpha = \beta \qquad \text{—\textit{weak antisymmetry}} \tag{14}$$

$$\alpha \le \beta \ \& \ \beta \le \gamma \ \Rightarrow \ \alpha \le \gamma \qquad \text{—}\textit{transitivity}. \tag{15}$$

Thus $\le$ is a partial ordering relation on any set of cardinals. As usual, the notation $\alpha < \beta$ is an abbreviation for $\alpha \le \beta \ \& \ \alpha \ne \beta$.

The concept of finiteness is more delicate than you might expect. The theorems in this paragraph are easy to establish, but others, seemingly no less elementary, depend on the axiom of choice, and are thus considered only in later notes. First, $\mathbb{N}$ *is infinite—that is, not finite*—for if there existed $n \in \mathbb{N}$ and a bijection $f : \{m \in \mathbb{N} : m < n\} \to \mathbb{N}$, then $n > 0$ and the function $e : \{m \in \mathbb{N} : m < n-1\} \to \mathbb{N}$ defined by setting

$$e(m) = \begin{cases} f(m) & \text{if } f(m) < f(n-1) \\ f(m)-1 & \text{otherwise} \end{cases}$$

would be bijective, so $\{m \in \mathbb{N} : m < n\} \approx \{m \in \mathbb{N} : m < n-1\}$, contradicting (9). Second,

$$E \preceq F \ \& \ F \ \textit{is finite} \ \Rightarrow \ E \ \textit{is finite}$$

for any sets E, F. To verify that, consider the minimum $n \in \mathbb{N}$ for which there exists an injection $f : E \to \{m \in \mathbb{N} : m < n\}$. Such an f must be bijective, for if not, there would exist $k \in \mathbb{N}$ such that $k < n$ and $k \notin \operatorname{Rng} f$, so that $n > 0$ and the function $e : E \to \{m \in \mathbb{N} : m < n-1\}$ defined by setting

$$e(m) = \begin{cases} f(m) & \text{if } f(m) < k \\ f(m)-1 & \text{otherwise} \end{cases}$$

would be injective, contradicting the minimality of n. Finally, this result clearly implies that *every subset of a finite set is finite.*

You can prove recursively that if X and Y are disjoint finite sets then $X \cup Y$ is finite and $\#(X \cup Y) = \#X + \#Y$ (16). Moreover, you can easily show that for all sets X, Y, X', and Y',

$$X \approx X' \ \& \ Y \approx Y' \ \& \ X \cap Y = \phi = X' \cap Y' \ \Rightarrow \ X \cup Y \approx X' \cup Y', \tag{17}$$

and if Y is a singleton, then $X \times Y \approx X$ (18). These results permit definition of the *sum*

$$\#X + \#Y = \#\big((X \times \{0\}) \cup (Y \times \{1\})\big)$$

of *any* cardinals $\#X$ and $\#Y$. It follows that $\#(X \cup Y) = \#X + \#Y$ for *any* disjoint sets X and Y. You can demonstrate the following laws easily by constructing various functions: for all cardinals α, β, and γ,

$$\alpha + \beta = \beta + \alpha \qquad \text{—}\textit{commutativity} \tag{19}$$
$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma \qquad \text{—}\textit{associativity} \tag{20}$$
$$\alpha + 0 = \alpha \tag{21}$$
$$\beta \le \gamma \ \Rightarrow \ \alpha + \beta \le \alpha + \gamma \qquad \text{—}\textit{monotonicity}. \tag{22}$$

The associativity equation permits the abbreviation $\alpha + \beta + \gamma$ for either side of it.

You can prove recursively that if X and Y are finite sets then $X \times Y$ is finite and $\#(X \times Y) = (\#X)(\#Y)$ (23). Moreover, you can easily show that for all sets X, Y, X', and Y',

$$X \approx X' \;\&\; Y \approx Y' \Rightarrow X \times Y \approx X' \times Y'. \tag{24}$$

These results permit the definition of the *product*

$$(\#X)(\#Y) = \#(X \times Y)$$

of *any* cardinals $\#X$ and $\#Y$. You can demonstrate the following laws easily by constructing various functions: for all cardinals α, β, and γ,

$$\alpha\beta = \beta\alpha \qquad \text{—\textit{commutativity}} \tag{25}$$
$$\alpha(\beta\gamma) = (\alpha\beta)\gamma \qquad \text{—\textit{associativity}} \tag{26}$$
$$\alpha 0 = 0 \;\&\; \alpha 1 = \alpha \tag{27}$$
$$\beta \le \gamma \Rightarrow \alpha\beta \le \alpha\gamma \qquad \text{—\textit{monotonicity}} \tag{28}$$
$$\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma \qquad \text{—\textit{distributivity}.} \tag{29}$$

The associativity equation permits the abbreviation $\alpha\beta\gamma$ for either side of it.

You can prove recursively that if X and Y are finite sets then X^Y is finite and $\#(X^Y) = \#X^{\#Y}$ (30). Moreover, you can easily show that for all sets X, Y, X', and Y',

$$X \approx X' \;\&\; Y \approx Y' \Rightarrow X^Y \approx X'^{Y'}. \tag{31}$$

These results permit the definition $\#X$ *raised to the power* $\#Y$ —

$$\#X^{\#Y} = \#(X^Y)$$

—for *any* cardinals $\#X$ and $\#Y$. You can demonstrate the following laws easily by constructing various functions: for all cardinals α, β, and γ,

$$\alpha^{\beta+\gamma} = \alpha^\beta\alpha^\gamma \qquad \alpha^{\beta\gamma} = (\alpha^\beta)^\gamma \tag{32, 33}$$
$$\alpha^0 = 1 \qquad \alpha^1 = \alpha \tag{34, 35}$$
$$\alpha \ne 0 \Rightarrow 0^\alpha = 0. \tag{36}$$

This discussion closes with an 1875 theorem of Georg Cantor: for any set X, $\#X < \#\mathcal{P}X = 2^{\#X}$. *Proof.* Define $\varphi : \#\mathcal{P}X \to \{0,1\}^X$ by setting

$$\varphi_Y(x) = \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{if } x \notin Y \end{cases}$$

for each $Y \subseteq X$ and each $x \in X$. Then φ is bijective, so $\#\mathcal{P}X = 2^{\#X}$. Now define $\chi : X \to \mathcal{P}X$ by setting $\chi(x) = \{x\}$ for each $x \in X$. Then χ is injective, so $\#X \le \#\mathcal{P}X$. Finally, suppose there existed a bijection $\psi : X \to \mathcal{P}X$. Let

$$Y = \{x \in X : x \notin \psi(x)\} \qquad y = \psi^{-1}(Y).$$

Then $y \in Y \Leftrightarrow y \notin \psi(y) = Y$, contradiction! Thus ψ cannot exist, so $\#X \ne \#\mathcal{P}X$.

Routine exercises

1. Prove all the numbered assertions in these notes.

2. Show that if X is a finite set and $f : X \to X$, then f is injective if and only if it's surjective.

Substantial problem

1. Prove Alfred Tarski's 1924 theorem that a set X is finite if and only if

$$\forall \mathcal{T}[\phi \neq \mathcal{T} \subseteq \mathcal{P}X \Rightarrow (\exists T \in \mathcal{T})[\mathcal{P}T \cap \mathcal{T} = \{T\}]].$$