AXIOM OF CHOICE

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Direct Product

If A is a function with nonempty domain I, then

$$\operatorname{Rng} A = \{A_i : i \in I\}$$

is called a *family of sets* A_i *indexed by* the members $i \in I$. A *choice function* for this family is a function $c : I \to \bigcup_i A_i$ such that $c_i \in A_i$ for each $i \in I$. The set

$$\mathop{\times}_{i \in I} A_i$$

of these choice functions is called the *direct product* of the family $\{A_i : i \in I\}$.

Axiom of Choice

Suppose $0 \neq A_i \subseteq \mathbb{N}$ for each $i \in I$. You can then construct a choice function $c \in \times_i A_i$ by setting c_i equal to the first element of A_i, for each $i \in I$. In general, however, no such construction method is available. Filling this need requires a basic principle of set theory known as the *axiom of choice*: *every indexed family of nonempty sets has a choice function*. That is,

$$\forall i\,[A_i \neq \phi] \;\Rightarrow\; \times_i A_i \neq \phi\,.$$

The following result is an alternative form of the axiom of choice, due to Paul Bernays in 1941 (see substantial problem 2): *for each relation* R *there exists a function* $F \subseteq R$ *such that* $\operatorname{Dom} F = \operatorname{Dom} R$. In fact, you can find

$$F \in \mathop{\times}_{i \in \operatorname{Dom} R} R[\{i\}]\,.$$

Left and Right Inverses

Let $f : X \to Y$ and $g : Y \to X$. Then g is called a *left inverse* of f if $g \circ f = I_X$, and a *right inverse* of f if $f \circ g = I_Y$.

The following result doesn't depend on the axiom of choice: *if* $X \neq \phi$ *then* f *is injective if and only if it has a left inverse*. *Proof*. If g is a left inverse of f, then $f(x) = f(x') \Rightarrow x = I_X(x) = g(f(x)) = g(f(x')) = I_X(x') = x'$, hence f is injective, so $\breve{f}$ is a function. Let $x_0 \in X$ and define $g : Y \to X$ by setting

$$g(y) = \begin{cases} \breve{f}(y) & \text{if } y \in \operatorname{Rng} f \\ x_0 & \text{otherwise} \end{cases} .$$

Then g is a left inverse of f.

The following complementary result is another alternative form of the axiom of choice: *f is surjective if and only if f has a right inverse*. *Proof*. If g is a right inverse of f, then for each $y \in Y$, $f(g(y)) = I_Y(y) = y$, hence f is surjective. (This part of the proof didn't depend on the axiom of choice.) Conversely, suppose f is surjective. By Bernays' form of the axiom of choice there's a function $g \subseteq \breve{f}$ such that $\operatorname{Dom} g = \operatorname{Dom} \breve{f} = \operatorname{Rng} f = Y$. Then g is a right inverse of f.

If f has a left inverse g and a right inverse g', then f is bijective and $g = g' = f^{-1}$. *Proof*. $g = g \circ I_Y = g \circ (f \circ g') = (g \circ f) \circ g' = I_X \circ g' = g'$; therefore, $g = g'$ is an inverse of f.

Trivial questions

1. $\exists i\ [\ A_i = \phi\ ] \Rightarrow \times_i A_i = ?$
2. $\forall i\ [\ A_i = B\ ] \Rightarrow \times_i A_i = ?$
3. $\forall i\ [\ A_i \subseteq B_i\ ] \Rightarrow$ what (about the corresponding direct products)?

Routine exercises

1. Prove that $I = \{0\} \Rightarrow \bigtimes_{i \in I} A_i \approx A_0$ and $I = \{0,1\} \Rightarrow \bigtimes_{i \in I} A_i \approx A_0 \times A_1$.

2. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is *continuous* at $x \in \mathbb{R}$ if and only if

   $$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x' \in \mathbb{R})\ [\ |x - x'| < \delta \Rightarrow |f(x) - f(x')| < \varepsilon\ ].$$

   Recall also that a sequence $c \in \mathbb{R}^{\mathbb{N}}$ converges to $x \in \mathbb{R}$ —written $\lim_n c_n = x$ —if and only if

   $$(\forall \varepsilon > 0)(\exists m \in \mathbb{N})(\forall n \in \mathbb{N})\ [\ m < n \Rightarrow |c_n - x| < \varepsilon\ ].$$

   Show that f is continuous at x if and only if for all sequences c, $\lim_n c_n = x$ implies $\lim_n f(c_n) = f(x)$. Show precisely where you used the axiom of choice in your proof.

3. Construct sets X and Y and functions f and f' from X to Y such that f has two left inverses but no right inverse and f' has two right inverses but no left inverse.

4. Show that for all nonempty sets X and Y, $X \preceq Y$ if and only if there exists a surjection from Y to X.

5. A function f from a set X to itself is called *idempotent* if $f \circ f = f$.
   a. Show that if g is also an idempotent function from X to itself, and $f \circ g = g \circ f$, then $f \circ g$ is idempotent.
   b. Show that the identity I_X is the only injective idempotent function from X to itself and the only surjective idempotent function from X to itself.
   c. Where did you use the axiom of choice in parts a and b?
   d. Use part b and substantial problem 3 of the "Basic Set Theory" notes to provide two alternative descriptions of I_X that don't refer to elements of X. Which is preferable?

Substantial problems

1. Consider an indexed family of sets $\{A_i : i \in I\}$. Let $B = \times_i A_i$. For each i, define $\pi_i : B \to A_i$ by setting $\pi_i(a) = a_i$ for each $a \in B$. Clearly, each π_i is surjective. Show that, given any set X, and any family of functions $f_i : X \to A_i$ for each $i \in I$, there exists a unique $g : X \to B$ such that $\pi_i \circ g = f_i$ for all $i \in I$.

   Let B' be a set and suppose, for each i, that $\pi'_i : B' \to A_i$ surjectively. Suppose that given any set X and any family of functions $f_i : X \to A_i$ you can find a unique $g : X \to B'$ such that $\pi'_i \circ g = f_i$ for all $i \in I$. Show that there exists a unique bijection $e : B \to B'$ such that $\pi'_i \circ e = \pi_i$ for all $i \in I$.

2. a. Show that Bernays' form of the Axiom of Choice implies the one given in these notes.
   b. Show that the axiom of choice is implied by the proposition that every surjection should have a right inverse. This result is due to Paul Bernays, 1941. Hint: let R be a relation; define $H : R \to \operatorname{Dom} R$ by setting $H(<x, y>) = x$.

3. a. Consider an indexed family $\{A<i,j> : <i,j> \in I \times J\}$. Prove that

$$\bigcap_i \bigcup_j A_{<i,j>} = \bigcup_{g \in J^I} \bigcap_i A_{<i,g_i>} .$$

   Formulate and prove a dual result.
   b. Prove that the distributive law formulated in part (a) implies the axiom of choice. Hint: prove that any surjection $f : X \to Y$ has a right inverse; consider

$$A_{<y,x>} = \begin{cases} \{0\} & \text{if } f(x) = y \\ 0 & \text{otherwise} \end{cases} .$$

4. Show that the following conditions on a partially ordered set $<X, \le>$ are equivalent:

   a. X has a nonempty subset with no minimal element
   b. $(\exists s \in X^{\mathbb{N}})(\forall n \in \mathbb{N})[s_{n+1} < s_n]$.

   Show precisely where you used the axiom of choice in your proof. Formulate a similar result about maximal elements.

5. Consider some indexed families $\{A_i : i \in I\}$ and $\{B_i : i \in I\}$.

   a. Suppose $\forall i\,[A_i \approx B_i]$ and $\forall i,j\,[A_i \cap A_j = \phi = B_i \cap B_j]$. Prove that $\bigcup_i A_i \approx \bigcup_i B_i$.
   b. Suppose $\forall i\,[A_i \approx B_i]$. Prove that $\mathsf{X}_i A_i \approx \mathsf{X}_i B_i$.
   c. Define $\Sigma_i \#A_i$ and $\Pi_i \#A_i$.
   d. Suppose $\#I = \alpha$ and $\forall i\,[\#A_i = \beta]$. Prove $\Sigma_i \#A_i = \alpha\beta$ and $\Pi_i \#A_i = \alpha^\beta$.

6. Let $\{\alpha_i : i \in I\}$ and $\{\beta_i : i \in I\}$ be indexed families of cardinals. Show that

   $$\forall i\,[\alpha_i < \beta_i] \Rightarrow \Sigma_i \alpha_i < \Pi_i \beta_i.$$

   Often called König's theorem, this result is in fact due to Zermelo [1908] 1970a. Show that it implies both the axiom of choice and Cantor's theorem about the cardinality of the power set.

References

Monk 1969.
Moore 1982; Smoryński's *MathSciNet* review of this is very informative.
Rubin and Rubin 1985.