Drafting the class outlines, I've been including suggestions for paper topics. Originally my strategy was to air them only after readings and discussions introduced the concepts involved. But the class is larger than I expected, and your interest in selecting paper topics early is stronger. To provide access to the suggestions without causing confusion when I edit the draft outlines, I'm collecting those suggestions in this document.

You may be searching for topics appropriate for a term paper in this or another course or for a master's expository paper. All mathematics professors should be able to suggest topics in their areas of greatest competency. For me, those are foundations of geometry, and some parts of logic, numerical analysis, software engineering, and history of mathematics. As possible projects occur to me during this course, I'll note them here as well as in the class outlines. In almost every case, further conversations would be required to clarify the scope and feasibility of a possible project. Please don't think that you need to select a project from *this* cumulative list. It is intended, rather, to suggest the *sort* of project that would be appropriate. The distinction between a term paper and a master's expository paper is vague; I won't always specify which type is suggested. A term paper must emphasize work in the foundations of mathematics. A master's paper need not, but should be deeper and broader in scope.

1. Describe and discuss some *alternatives to Kuratowski's definition of ordered pair*. I think a very young Norbert Wiener proposed one a few years before Kuratowski. Was it the first? How were ordered pairs handled before such a definition? Harris 1970 considers other alternatives, and points to another paper.

2. The terms "injection", "surjection", and "bijection" stem in part from *Franco-German rivalry* in mathematics and other areas, and thus open the door to a general topic: that rivalry. This would be a good project but not for a term paper in this course.

3. Kuratowski's work stems from *Polish nationalism*, specifically in mathematics, and thus opens the door to that topic. Some background literature is in English. Polish mathematical research was often published in French, and concentrated heavily on foundations of mathematics. Polish itself is probably not required for this project.

4. Compare and contrast, in critical detail, at least two methods of *constructing the real numbers* from the rationals or the natural numbers and proving that they form a complete ordered field. Many textbooks cover a single method in detail; a paper should be significantly broader than that, but might skip some details, as long as it says what is skipped. I don't think I've seen Weierstrass's method in any text-

book; but enough information may be available to include it in a project. Lightstone 1965 describes an interesting alternative to these methods.

5. *Category theory* is a fundamental subject in advanced algebra. Basing it on the set theory presented in this course requires limiting its strength inconveniently. Solomon Feferman recently presented a colloquium at Berkeley on strengthening set theory to accommodate category theory. That involves providing some features analogous to those of a "universal set" while avoiding contradictions analogous to Russell's. For someone already quite familiar with category theory, an investigation of this problem area and current research could result in a term paper. A more comprehensive study could lead to a master's expository paper.

6. The theory presented informally in the "Basic set theory" unit will be incorporated into Zermelo–Fraenkel set theory (ZF) later, after the formal syntax of first-order logic is introduced. Some features of this theory may seem out of step with logical methods you learned in elementary courses—particularly its insistence that no set should be "universal," containing all objects. Many texts employ that notion in developing simple Boolean logic. These notes don't, because ZF avoids it to forestall Russell's antinomy. Logicians have investigated alternative *formal set theories that do include a universal set*: they employ different means to avoid the antinomy. Recently, interest has increased in one of them, called NFU. The unpublished manuscript Holmes [1998] 2007 contains a presentation of NFU analogous to this course's exposition of ZF. A thorough comparison of the elementary parts of the two theories could constitute an excellent term paper. It would be a good choice for someone who has already studied elementary logic.

7. Kosmák 1980 presents many *extensions of exercises in the* Equivalences and Partitions *unit and related elementary results*. Some of those could form the core of a paper about equivalence relations. He didn't give many applications, but an expository paper should also show how the elementary theory of equivalences finds applications in other areas. It should be possible to unearth references to that. I have copies of several French papers by Paul Dubreil and Marie-Louise Dubreil-Jacotin that may be appropriate. Finberg et al. 1996 is intriguing and understandable.

8. Some information on the *project described in the* Partially Ordered Sets *unit* is recorded in the cited literature. Bjarni Jónsson has published some papers on that subject.

9. Describe and implement the algorithm in Flanders 1987 for generating the *transitive closure of a Boolean matrix*. Is it useful for counting isomorphism classes of four-, five- and six-element partially ordered sets?

10. Reporting progress on the question, *how many isomorphism classes of finite n-element partially ordered sets are there* could constitute a master's expository paper. I have some references.

11. I have many reprints of or references to papers, mostly in English, that pursue by elementary methods other *easily understandable questions about partially ordered sets*. I suspect that some could lead to master's projects in combinatorics or algebra. Galvin 1994 seems particularly promising.

12. E. H. Moore's relationship to Germany, his founding of the Chicago department, the work of the *American postulate theorists*, and Moore's work in analysis would make a great survey for a paper in the history course. For the foundations course, make it narrower but deeper, emphasizing the postulate theorists.

13. Pursue more details of the material in Tarski 1955, "*A lattice-theoretical fixpoint theorem and its applications*." I've recently been finding related material, applied to computer science and economics, that is quite elementary and could contribute to such a project, but I'm not organized enough yet to report it here.

14. Report more details about the *Cantor–Bernstein theorem*, for example: various proofs, faulty arguments, related theorems, and applications. You can get started by consulting references cited in these notes and using JSTOR to search the full text of various journals for `(Cantor OR Schroeder) AND Bernstein`. A related project is discussed later.

15. There is some accessible literature about the evolution of *Cantor's diagonal argument* and its relation to the argument that leads to Russell's antinomy. A report on that might make a term paper.

16. For each $n \in \mathbf{N}$, let AC_n stand for the statement *every indexed family of n-element sets has a choice function*. Interrelationships among these statements has been studied in detail using methods of set theory, combinatorics, and algebra. Years ago I attended a marvelous series of lectures on this subject by John Horton Conway, published in 1973. (The containing publication Mathias and Rogers 1973, is hard to locate, but available through our library.) Those, along with Sierpinski 1958, §VI.5, are a starting place. There is probably more ample research literature.

17. Using the accessible literature, investigate further *uses of the axiom of choice in real analysis*.

18. Investigate *extensions of the Cantor–Bernstein theorem*. In particular, to get the theorem of Yente the Matchmaker in Rosenholtz 2000 (temporarily online) is it merely necessary to require that the given injections be subsets of a given relation R and its converse, and to note that the constructed bijection is then included in

R? Is Rosenholtz's discussion of constructiveness equivalent to that in Sierpinski 1958, §2.6 exercise 2? Is Rosenholtz's title misleading? Does Yente's theorem occur in Tarski 1955, §3, Tarski [1928] 1930, Knaster and Tarski 1927, Tarski 1927, or Banach 1924? (Only very brief, easily translatable passages of those early papers should be relevant; I have copies of those that are not online.) These matters are discussed to some extent in Kanamori 1997 and in some detail in Wagon 1985, chapter 3. Finally, there are a number of "marriage theorems" in the combinatorics literature. To be sure, they're concerned only with finite sets, but are their proofs related to those considered in these other sources? I don't know the answers to any of these questions, but would like to.

19. Wagon 1985 is about the *Banach–Tarski "paradox,"* one of the most spectacular and counterintuitive consequences of the axiom of choice. At least one other, comparable, book is available, too. A report on this subject would be intriguing. It would probably require background in advanced real analysis.

20. Investigate further and report on the *controversy stimulated by Zermelo [1904] 1970*. Most of the original materials are now available in English.

21. *The ultrafilter theorem leads to a representation theorem*: every distributive lattice is isomorphic to a lattice of sets. There is a considerable literature connecting these theorems, similar ones for other algebraic structures such as Boolean algebras, representations as lattices of particular kinds of sets or functions, various forms of the axiom of choice, and various major theorems in general topology and functional analysis. Most of that literature is in English, and is at the same level as the solution of substantial problem 8. This work started with that of Marshall Stone and Garrett Birkhoff in the 1930s. I suspect that several projects could be fashioned from a survey of some of that literature.

22. Kanamori 1997, 305, refers to versions of Bourbaki's fixpoint theorem used by Dana Scott and others to develop *denotational semantics for programming languages*. This is intriguing: Scott was one of my teachers, both in logic and computer science, but before he did that work. I've never followed up on it, and would welcome someone's explaining it to me.

23. I'd not encountered *Kanamori's work* before preparing for this class. Several of his recent expository papers on set theory are reviewed in MR. Perhaps one of them could provide leads for a project.

24. Substantial problems 10–11 in the "Cardinals II" unit could spur an investigation of the *cardinal properties of various sets and functions encountered in real analysis*. See also Gödel 1947. There is a large literature on this subject.

25. Substantial problems 12–13 in the "Cardinals II" unit could spur an investigation of the *generalized continuum hypothesis* or a study of *advanced ordinal arithmetic*. For the former, see Gillman 2002. There are large literatures on these subjects.