

1. The first part of this lecture was devoted to quiz 2.
2. *Perspective drawing*
 - a. Points at infinity
 - i. Consider a scene consisting of two lines g, h on flat ground near a painter, not necessarily parallel. Suppose he's drawing them on a flat screen ε placed vertically on an easel in front of him. Join his eye I to g, h with two planes γ, η . To the painter, lines g, h appear the same as the intersections g', h' of planes γ, η with the screen ε . Call g', h' the *images* of g, h . Note: *the image of a straight line is a straight line*.
 - ii. Given a line k' on the screen, you can usually find on the ground exactly one line k of which it is the image: join k' to the eye I with a plane κ and consider the intersection k of κ with the ground. But one line on the screen—the horizontal line at eye level, called the *horizon*—is an exception, because the plane joining the horizon to the eye I does not intersect the ground.
 - iii. Now suppose the lines g, h on the ground are parallel. Then the line $f \parallel g, h$ through the eye I lies on both planes γ, η : it is their intersection. This line f intersects the screen ε at a point P' that lies on both images g', h' . Note: *parallel lines on the ground don't intersect, but their images intersect on the screen somewhere on the horizon*.
 - iv. Any line l on the floor parallel to g, h is also parallel to f ; therefore, f lies on the plane through l and the eye I and the image l' must pass through the same point P' on the horizon. Note: *the images of any set of mutually parallel lines on the ground pass through the same point on the horizon. Images of lines on the ground not parallel to these pass through different points on the horizon*.
 - v. Now imagine the plane of the ground extended by adding new *ideal points* P of which the points P' can be regarded as images. Such a point P should be regarded as lying on all lines l on the ground that are parallel to some fixed line g . The set of all such lines l is called a *parallel pencil*, and P is the ideal point corresponding to that pencil. Finally, extend your concept of the ground plane by regarding all those ideal points as lying on a new *ideal line* h_∞ . Note: *in the extended plane, any two distinct points, ideal or not, lie on exactly one line, which may be ideal; and any two distinct lines, ideal or not, intersect in exactly one point, which may be ideal*.
 - vi. These ideal points are often called *points at infinity*; the ideal line h_∞ , the *line at infinity*; and the extended ground plane, the *projective plane*.
 - vii. In many ways, the geometry of the projective plane is simpler than that of the Euclidean plane, because we needn't treat intersecting and parallel pairs of lines differently. Mathematicians gradually developed facility

with this technique, and by the early 1800s had developed a projective counterpart of Cartesian coordinates and analytic geometry: *homogeneous* coordinates and *projective* geometry. Surprisingly, although much of today's geometrical research is conducted in that context, projective geometry has disappeared from university curricula.

- viii. Much of this discussion can be reformulated for three-dimensional geometry (and higher dimensions), but we lose the artistic motivation.

We do that because it makes higher-dimensional geometry easier.

- b. The German Albrecht Dürer studied with the Italians, and became particularly virtuosic. Click [here](#) for his etching showing one of his techniques, and for his picture of St. Jerome translating the Bible. The former is a page from his 1525 book about this kind of drawing. Mathematicians have verified that all perspective details of the St. Jerome picture are correct, except the size of the lion. Dürer evidently knew that correct perspective would seem unrealistic to the viewer: our eyes, in motion, compensate for the nearness. Therefore Dürer drew the lion smaller than perspective mathematics would prescribe.
- c. I showed in class a lithograph of Dürer's house in Nürnberg.
- d. Click [here](#) for a humorously incorrect perspective drawing that I saw fifteen years ago in Napier, New Zealand. (It stood along a sidewalk, hiding from view the construction of a park, and showing it would eventually look like.)

3. *Struik, sections 5.8–5.9*

- a. Struik pointed out that in the late 1500s applied mathematics was expanding from the commercial realm to projects of longer duration such as public works and astronomy.
- b. Astronomical work and surveying required accurate trigonometric tables. Here's a sketch of how that work proceeded. The Alexandrian Greeks could use Pythagoras' theorem and similar triangles to find the trigonometric functions of $45^\circ, 36^\circ, 30^\circ$, and they could use difference formulas to get the functions of $9^\circ, 6^\circ, 3^\circ$. You can't get to 1° that way, but they knew how to use half-angle formulas to get the function values for 1.5° and 0.75° . They had also discovered that the function $\sin \theta / \theta$ for acute angles θ is decreasing. (Of course, they wouldn't have expressed it that way.) This leads to the inequalities $(\sin 3/4^\circ)/3/4 > \sin 1^\circ > (\sin 1\frac{1}{2}^\circ)/1\frac{1}{2}$. Shortcutting the Greeks' computation by using software to compute the sines, I can get $\sin 1^\circ$ correct to 5 decimal places from this formula. Greater accuracy would come from doing more trigonometric manipulations to get a thinner sandwich. Once you get the functions of 1° you can use the addition formulas to fill in the tables at 1° intervals. For finer spacing, use more trigonometry.