Thomsen's Equation

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1. INTRODUCTION. In plane Euclidean geometry reflections across lines are fundamental isometries. Every isometry is a composition of at most three such reflections. Compositions of two reflections are rotations or translations. Compositions of three are glide reflections. (For a thorough introduction to this area of geometry, see Smith [**8**, chap. 6].) This paper is about the reflections across edge lines a, b, and c opposite corresponding vertices of a triangle ΔABC. Since the lines are rarely mentioned again, notation is simplified by letting a, b, and c denote the reflections, too. Composition of plane motions, such as reflections, is indicated by juxtaposition, with the right-hand motion performed first. In this paper, describing a motion as a composition of reflections is just as important as determining its effect. Therefore a juxtaposition ba or abc, for example, will indicate a particular sequence of reflections as well as the resultant motion, which in these cases is a rotation or a glide reflection. The context will determine whether such a string denotes a sequence of letters, a sequence of reflections, or the resultant motion.

Compositions abc and bca are glide reflections, so their squares are translations. Since translations commute,

$$(abc)^2(bca)^2 = (bca)^2(abc)^2.$$

Writing this out and using equations

$$aa = bb = cc = \iota \text{ (the identity motion)} \tag{1}$$

to move all the reflections to the left-hand side of the equation yields

$$abcabcbcabcacbacbcbacb = \iota. \tag{2}$$

This equation with twenty-two reflections on the left holds for every triangle. *Is there a shorter such equation?*

Equations that can be derived from (1) alone, such as $aabb = \iota$ and $abba = \iota$, are called *trivial*. This paper shows that *no nontrivial equations shorter than* (2) hold for all triangles and demonstrates how to derive *all* such generally valid equations.

Gerhard Thomsen posed these questions in 1931 in the problem section of the *Jahresbericht der Deutschen Mathematiker Vereinigung*. Hellmuth Kneser answered them the same year [**6**]. Kneser presented a set of equations, including (2) but necessarily others too, from which *all* generally valid equations can be derived. Most of the paper at hand is based on Kneser's solution, a marvel of ultraconcise exposition. For the derivability result, he employed the notion of algebraic independence of complex numbers. To show that (2) is the shortest nontrivial generally valid equation, he made striking use of the familiar hexagonal "honeycomb" lattice.

The present paper stems from the conviction that this elegant mathematics and its context deserve to be better known. Sections 2 and 3 present these results in detail.

Completely rigorous arguments would require some exhaustive descriptions of complicated algorithms, along with unwieldy proofs that they always work. This expository paper avoids that level of detail by presenting examples—one via *Mathematica*— sufficiently general that one can discern the patterns and, after following them by hand, see why the observed patterns always occur. Section 4 shows that there are no corresponding results for higher dimensions. Section 5 places Kneser's and Thomsen's work in historical context.

2. DERIVING GENERALLY VALID EQUATIONS. This section shows that if a composition of reflections a, b, and c is equal to the identity for every triangle, then one can deduce that fact from equations (1) and certain equations that involve the rotations

$$p = bc \qquad q = ca$$

with centers A and B, respectively. Theorem 1 shows how to reformulate in terms of p and q many questions about compositions of reflections. The reader can supply its straightforward proof.

Theorem 1. *Let Φ be a composition of n reflections selected from the list $\{a,b,c\}$. If $\Phi = \iota$ for some triangle, then n is even. When n is even, the following table will convert Φ to a composition Ψ of rotations involving only p and q such that the equation $\Phi = \Psi$ holds for any triangle:*

$$\begin{array}{lll} aa = \iota & ba = pq & ca = q \\ ab = q^{-1}p^{-1} & bb = \iota & cb = p^{-1} \\ ac = q^{-1} & bc = p & cc = \iota. \end{array}$$

It's convenient to identify points with complex numbers in the standard way. Direct (sense-preserving) motions of the complex plane then take the algebraic form $z \to uz + w$ for some point u on the unit circle U and some complex number w. Such a mapping will be denoted by $[u,w]$. A mapping $[u,0] : z \to uz$ is a rotation about the origin through angle $\arg u$, and $[1,0]$ is the identity mapping ι. Any two such rotations commute. The mapping $[1,w] : z \to z + w$ is the translation by the vector w. Any two translations commute, but translations and rotations don't, in general. Each direct motion is the composition of a rotation about the origin and a translation. Composition of direct motions follows the rule

$$[u,w][v,x] = [uv, ux + w]$$

because it maps z to

$$u(vz + x) + w = (uv)z + (ux + w).$$

The composition is also a direct motion: uv lies on U. The next theorem follows directly from the composition rule.

Theorem 2. *For any $u, v \in U$ with $v \neq 1$, a composition of powers of the rotations $p = [u, 0]$ and $q = [v, 1]$ has algebraic form $\Phi = [u^m v^n, w]$, where m and n are the sums of the exponents of p and q, respectively, that occur in the composition and w is some complex number.*

This paragraph and the following one show how to use complex coordinates to select ΔABC in a particular way for the next theorem. First, let u be any transcendental point on U. (Only countably many points on U are algebraic, so some must be transcendental.) Next, select a point v on U so that u and v are algebraically independent over the rational field $\mathbb{Q}$: that is, the pair (u, v) is not a root of any nonzero polynomial in two variables with rational coefficients. To do that, note that the values $f(u)$ of polynomials f in one variable with rational coefficients constitute a countable set $\mathbb{Q}[u]$. The set P of all single-variable polynomials with coefficients in $\mathbb{Q}[u]$ is therefore also countable. Since a polynomial has only finitely many roots, the set R of all roots of polynomials in P is countable. Any point v in $U - R$ then satisfies the algebraic independence requirement. As a consequence, $u^m v^n = 1$ for integers m and n only if $m = n = 0$. Cases of this result with $m, n = 0, 1$ are used in the next paragraph; its full force is used for Theorem 3.

Now let A be the origin, $p = [u, 0]$, and $q = [v, 1]$. Then p is a rotation about A, and q is a rotation about the unique solution B of the fixpoint equation $B = q(B) = vB + 1$. (By algebraic independence, $v \neq 1$.) Moreover, $A \neq B$ because $q(A) \neq A$. Let c be the (reflection across the) line joining A and B. Every plane rotation is the composition of reflections across two lines through its center, either of which can be selected arbitrarily, hence $p = bc$ and $q = ca$ for (the reflections across) some lines b and a through A and B, respectively. Therefore $ba = bcca = pq = [u, 0][v, 1] = [uv, u]$ is a rotation about the unique solution C of the fixpoint equation $C = pq(C) = uvC + u$, and a and b are distinct lines intersecting at C. It’s easy to check that C doesn’t lie on c, hence A, B, and C form a triangle. The following result follows from Theorem 2 and the algebraic independence of u and v.

Theorem 3. *If ΔABC is constructed as in the previous paragraph, if Φ, m, and n are defined as in Theorem 2, and if $\Phi = \iota$, then $m = n = 0$.*

According to the next theorem, it’s possible to express compositions such as Φ in terms of special commutators—mappings of the form

$$K(m, n) = p^m q^n p^{-m} q^{-n}.$$

for integers m and n.

Theorem 4. *If the sums of the exponents of p and q in a composition Φ of powers of p and q are both zero, then Φ can be rewritten as a composition of special commutators and their inverses.*

Sketch of proof: It is a straightforward computation to verify the equation

$$p^k q^l p^m q^n = K(k,l)K(k+m,l)^{-1}p^{k+m}q^{l+n}.$$

This is employed to “reorganize” Φ as in the following example:

$$\begin{aligned}(p^3q^4p^{-1}q^{-2})p^{-2}q^{-2} &= (K(3,4)K(2,4)^{-1}p^2q^2)p^{-2}q^{-2}\\ &= K(3,4)K(2,4)^{-1}K(2,2).\end{aligned}$$

For longer compositions one continues this process until reaching a step with $k+m = l+n = 0$. To handle a composition beginning with a power of q, prefix p^0. A formal proof of this theorem could proceed by induction on the number of terms in Φ. ■

Before analyzing such compositions further, one must see how these commutators are expressed in terms of u and v. This is best achieved by using software to examine sufficiently complicated examples, then following the computation by hand to see why the pattern persists. Without software, a tedious hand calculation is probably unavoidable.

Theorem 5. *For any integers m and n,*

$$K(m,n) = \begin{cases}[1,(u^m-1)(v^{n-1}+\cdots+v+1)] & \textit{if } n>0,\\ [1,0] & \textit{if } n=0,\\ [1,(1-u^m)(v^n+\cdots+v^{-2}+v^{-1})] & \textit{if } n<0.\end{cases}$$

These commutators are translations, hence they commute.

Sketch of proof: The following *Mathematica* commands define (noncommutative) composition, inverse, and power operations on objects `dm[u,v]` corresponding to direct motions $[u,v]$, and define a function whose values `K[m,n]` represent commutators:

```
dm[u_, v_] ** dm[w_, x_] ^:= dm[u w, u x + v];
inv[dm[u_, v_]] := dm[1/u, -v/u];
dm[u_,v_]^n_ ^:= If[n < 0, inv[dm[u,v]^(-n)],
                    If[n == 0, dm[1,0],
                      dm[u,v] ** dm[u,v]^(n - 1)]];
```

```
Clear[u,v];
p = dm[u,0]; q = dm[v,1];
K[m_,n_] := Simplify[p^m ** q^n ** p^(-m) ** q^(-n)];
```

The reader is invited to execute `K[m,n]` for `m`, `n` $= \pm 6, \pm 7$ and other arbitrarily selected values to see the patterns noted. ■

Next, compositions of powers of these special commutators are analyzed.

Theorem 6. *If a composition*

$$\Phi = \prod_{i=1}^{l} K(m_i, n_i)^{e_i}$$

is equal to the identity ι *for the selected* ΔABC, *then for any integers* m *and* n *the sum of the exponents* e_i *corresponding to the commutators with* $m_i = m$ *and* $n_i = n$ *is zero.*

Sketch of proof: Let

$$w_i = \begin{cases} \left(u^{m_i} - 1\right)\left(v^{n_i - 1} + \cdots + v + 1\right) & \text{if } n_i > 0, \\ \left(u^{m_i} - 1\right)\left(v^{n_i} + \cdots + v^{-2} + v^{-1}\right) & \text{if } n_i < 0. \end{cases}$$

Then

$$\Phi = \prod_{i=1}^{l} [1, e_i w_i] = \left[1, \sum_{n_i > 0} e_i w_i - \sum_{n_i < 0} e_i w_i\right].$$

If $\Phi = \iota$, then

$$\sum_{n_i > 0} e_i w_i - \sum_{n_i < 0} e_i w_i = 0.$$

The conclusion of the theorem is obtained by writing the left-hand side of the previous equation as a polynomial in u and v. That’s best seen by considering an example:

i	m_i	n_i	i	m_i	n_i
1	3	3	5	2	–2
2	3	2	6	2	–1
3	3	2	7	2	–1
4	3	1			

$$0 = \sum_{n_i > 0} e_i w_i - \sum_{n_i < 0} e_i w_i$$

$$\begin{aligned}
&= e_1(u^3-1)(v^2+v+1) + e_2(u^3-1)(v+1) + e_3(u^3-1)(v+1) \\
&\quad + e_4(u^3-1) - e_5(u^2-1)(v^{-2}+v^{-1}+1) - e_6(u^2-1)(v^{-1}+1) \\
&\quad - e_7(u^2-1)(v^{-1}+1) \\
&= e_1u^3v^2 + (e_1+e_2+e_3)u^3v + (e_1+e_2+e_3+e_4)u^3 - e_1v^2 \\
&\quad - (e_1+e_2+e_3)v - e_5u^2v^{-2} - (e_5+e_6+e_7)u^2v^{-1} \\
&\quad - (e_5+e_6+e_7)u^2 + e_5v^{-2} + (e_5+e_6+e_7)v^{-1} \\
&\quad + (-e_1-e_2-e_3-e_4+e_5+e_6+e_7).
\end{aligned}$$

The algebraic independence of u and v permits derivation of the following equations from the first two lines of the previous formula:

$$\begin{aligned}
&e_1 = 0 && e_5 = 0 \\
&e_1+e_2+e_3 = 0 && e_5+e_6+e_7 = 0. \\
&e_1+e_2+e_3+e_4 = 0 &&
\end{aligned}$$

In turn, these yield

$$\begin{aligned}
0 = e_1 &= \textstyle\sum\{e_i : m_i = 3,\ n_i = 3\} \\
0 = e_2 + e_3 &= \textstyle\sum\{e_i : m_i = 3,\ n_i = 2\} \\
0 = e_4 &= \textstyle\sum\{e_i : m_i = 3,\ n_i = 1\} \\
0 = e_5 &= \textstyle\sum\{e_i : m_i = 2,\ n_i = -2\} \\
0 = e_6 + e_7 &= \textstyle\sum\{e_i : m_i = 2,\ n_i = -1\}.
\end{aligned}$$

The pattern of this example persists in general. A formal proof would have to include a complete and precise definition for the general case of the preceding three systems of equations, and algorithms for deriving the second from the first and the third from the second. ■

Theorem 7. *If a composition of reflections selected from the list* $\{a,b,c\}$ *is equal to the identity for every triangle, then this fact is derivable from equations* (1) *and the equations*

$$K(k,l)K(m,n) = K(m,n)K(k,l) \tag{3}$$

for all k, l, m, *and* n, *which express commutativity of the commutators.*

Sketch of proof: Choose ΔABC with u and v algebraically independent. Convert the composition of the given sequence of reflections into one of powers of p and q, then into a composition of powers of commutators. For example, consider

$$\Phi = K(1,2)^2K(4,5)K(1,2)^{-2}K(4,5)^{-1}.$$

By Theorem 6 and the commutativity of commutators this can be rearranged so that the result appears to be a triviality:

$$\Phi = K(1,2)^2K(1,2)^{-2}K(4,5)K(4,5)^{-1}.$$

It’s not quite so, however. After this equation is written in terms of p and q, then in terms of the original a, b, and c, equations $aa = bb = cc = \iota$ must be applied many times to reduce the right-hand side to the identity. For the preceding example, this final process starts with the step

$$K(1,2) = pq^2p^{-1}q^{-2} = (bc)(caca)(cb)(acac) = bacacbacac. \qquad \blacksquare$$

3. FINDING THE SHORTEST GENERALLY VALID EQUATION. This section shows that any equation shorter than (2) that can be derived from equations (1) and (3) must be derivable solely from equations (1). The argument is based on the characterization of derivability in Section 2 and on the regular hexagonal plane lattice, a portion of which is shown in Figure 1. Regard the lattice as fixed in the plane, with points X and Y assigned specific coordinates. Designate cell edges with letters a, b, and c according to their slopes, as indicated in the figure. (At first, ignore the arrows. They indicate vectors that are introduced later.)

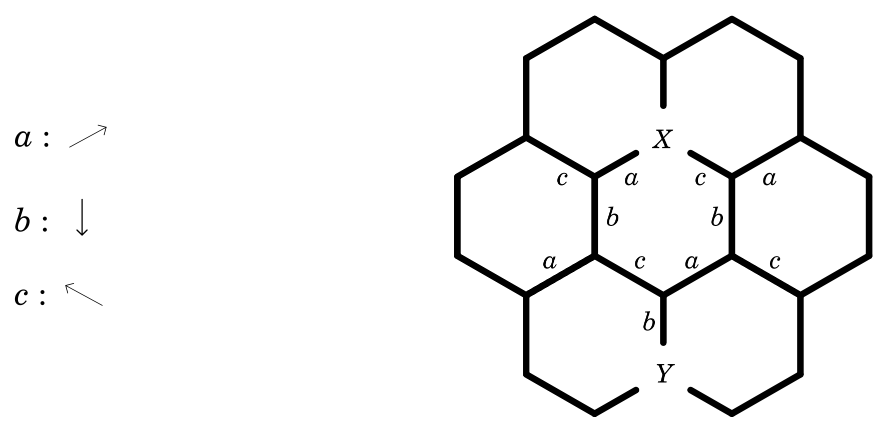

Figure 1. Hexagonal plane lattice.

A path φ along cell edges can be described completely by specifying its initial vertex and a finite sequence of letters selected from the list $\{a, b, c\}$ that indicates which edge to follow from each vertex. For example, the shortest paths from X to Y in Figure 1 are $\varphi = abcb$ and $\chi = cbab$. This discussion considers only *irredundant* paths: those that never traverse the same edge successively in opposite directions. The sequences of letters that describe them correspond to compositions of reflections a, b, and c considered earlier, simplified as far as possible by applying equations (1). The empty sequence corresponding to the identity motion describes trivial paths that never leave their initial vertices.

Paths φ and χ can be *concatenated* as follows. From the initial vertex of φ traverse in order the edges of φ, then edges that are parallel to the successive edges of χ. (The initial vertex of χ is irrelevant.) If the final edge of φ and the initial edge of χ are designated by the same letter, so that one would be traversing the same edge successively in opposite directions, omit both. Continue eliminating such pairs until an irredundant or trivial path is obtained; that is the concatenation $\varphi\chi$. The *reverse* of a path φ is the path φ^{-1} whose starting vertex is the final one of φ and whose edges are those of φ, traversed in the opposite order. The reverse of a trivial path is trivial, and the concatenation of a path and its reverse is trivial.

Theorem 8. *Concatenation and reversal of paths in the hexagonal lattice correspond to composition and inversion of the associated sequences of reflections, simplified according to equations* (1).

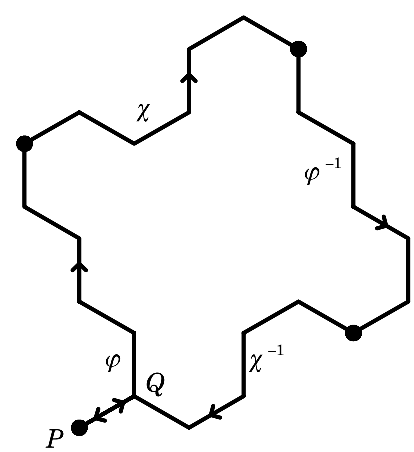

Figure 2. Lattice parallelogram $\varphi\chi\varphi^{-1}\chi^{-1}$.

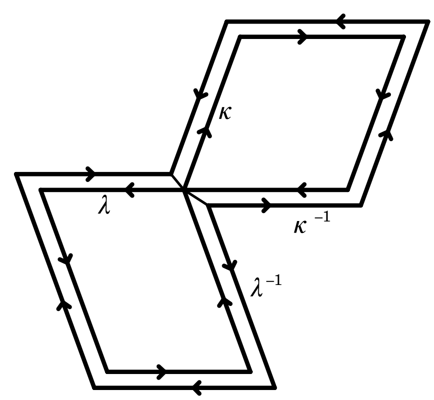

Figure 3. Path $\kappa\lambda\kappa^{-1}\lambda^{-1}$ corresponding to equation (4).

The path $\kappa = \varphi\chi\varphi^{-1}\chi^{-1}$ is called the *commutator* of paths φ and χ. Its associated motion is the commutator of those associated with φ and χ. Figure 2 is an example. (The arrows in the figure indicate only the order of the edges in the paths. The edges themselves are not directed. The lower left edge occurs twice, as initial segment of φ and final segment of χ^{-1}.) In this example, κ is closed. But that's not always the case: the reader should check another example in which χ has an odd number of edges! The next theorem explains the situation.

Theorem 9. *If paths φ and χ have even numbers of edges, then their commutator is a closed path.*

Sketch of proof: In this discussion, interpret a, b, and c not only as reflections and as lattice edges, but also as vectors, all with tails at the same lattice vertex and heads pointing away from it in the directions indicated in Figure 1. Let P be the initial vertex of φ. The vectors from P to the other vertices of φ are alternating $+/-$ sums of its edge vectors, starting with $+$ or $-$ depending on how φ leaves P. In Figure 2, for example, φ corresponds to the motion $abcbcb$ and its terminal vertex is $P + a - b + c - b + c - b$. (In contrast, consider a path ψ starting at the second vertex Q of φ and proceeding along edge b, then c. The vector from the initial vertex Q of ψ to its third vertex is $-b + c$ —the alternating sum starts with $-$.) In the alternating sum for the vector from P to the terminal vertex of χ, the part corresponding to φ^{-1} cancels that corresponding to φ if an even number of terms intervene—that is, if χ has an even number of edges. Similarly, the sums for χ and χ^{-1} cancel if φ has an even number of edges. ∎

If paths φ and χ have even numbers of edges, as in Figure 2, then their commutator is called a *lattice parallelogram*.

Equations (3) state the commutativity of commutators of powers of the rotations p and q. According to Theorem 7, they play an important role in characterizing the valid equations involving compositions of reflections a, b, and c. Equations (3) are equivalent to asserting that

$$K(k,l)K(m,n)K(k,l)^{-1}K(m,n)^{-1} = \iota \tag{4}$$

for all integers k, l, m, and n. Thus it's helpful to consider a path $\kappa\lambda\kappa^{-1}\lambda^{-1} = \mu$ corresponding to the left-hand side of equation (4). Figure 3 is a schematic example. Essentially, μ consists of a succession of four lattice parallelograms with a common vertex, the latter two being reversals of the first two.

Theorem 10. *The path μ corresponding to the left-hand side of* (4) *is closed, and its winding number with respect to any cell in the lattice is zero.*

Proof: Since κ and λ are closed, the part of μ that corresponds to κ^{-1} consists of the same edges as κ, but traversed backward. Thus any cell surrounded j times by κ will be surrounded j times in the opposite sense by κ^{-1}, hence must have winding number 0 with respect to μ. ∎

Theorem 11. *Let Φ be the composition of a finite sequence of reflections selected from the list $\{a,b,c\}$, with corresponding path φ. If $\Phi = \iota$ for all triangles, then the steps in deriving this fact from equations* (1) *and* (3) *correspond to operations on paths:*

inserting or removing edges traversed successively in opposite directions, and inserting or removing various four-parallelogram paths μ. *These operations have no effect on winding number and transform closed paths into closed paths. Thus* φ *must be closed and must have the same winding number with respect to any cell in the lattice as a trivial path, namely, zero.*

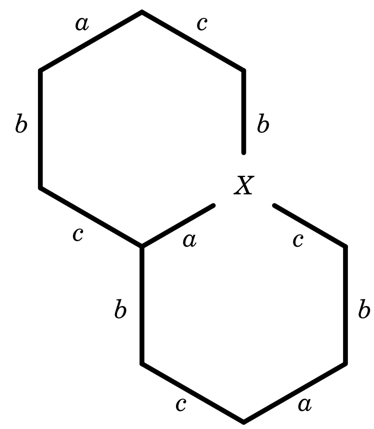

Figure 4. Hexagonal grid path corresponding to Thomsen's equation $(abcabc)(bcabca)(cbacb)(cbacb) = \iota$.

Finding a shortest universally valid equation $\Phi = \iota$ that is not derivable merely from equations (1) requires finding the shortest nontrivial path φ that satisfies the conditions in Theorem 11. Since φ must surround at least one cell once in each sense without traversing the same edge successively in opposite directions, it must in fact surround at least two cells in this manner. Figure 4 shows a shortest such path φ: starting at X, surround the bottom hexagon first counterclockwise $(abcabc)$, then the top one counterclockwise $(bcabca)$, followed in the opposite sense by the bottom $(cbacb)$, and finally the top $(cbacb)$. Composing the corresponding sequences of reflections yields the left-hand side of equation (2). Removing this four-parallelogram path from φ results in the trivial path, corresponding to the identity ι, the right-hand side of (2).

The two hexagons could be arranged to share an edge b or c instead of a, which would yield variants of equation (2) with a, b, and c permuted. These three equations with twenty-two terms on the left are thus the shortest that are valid for all triangles, save those equations derivable from equations (1) alone.

Kneser also showed that although all such valid equations can be derived from equations (1) and (3), equations (1) and the twenty-two term case (2) and its permuted variants don't by themselves form a basis for deriving all valid equations.

4. HIGHER DIMENSIONS. In 1931 Thomsen and Kneser didn’t consider analogous higher-dimensional questions, though they are quite natural (see Section 5). There are more types of reflections in higher-dimensional geometries. What *are* the analogous questions? For most considerations having to do with rigid motions, the roles of lines in plane geometry are played by planes and hyperplanes in three and more dimensions. For example, in plane, solid, or n-dimensional geometry, every motion is a composition of at most three, four, or $n+1$ reflections across lines, planes, or hyperplanes, respectively. (For the three-dimensional theory, see Smith [**8**, chap. 7].)

First, consider three dimensions. Let Φ be a finite sequence of reflections selected from the list $\{a,b,c,d\}$ whose composition is the identity ι for every choice of these corresponding planes. It’s easy to see that, if at most three different reflections occur in Φ, then repeated application of equations $aa = bb = cc = dd = \iota$ must reduce the composition to the identity. To that end, assume—without loss of generality—that a occurs first in Φ and d doesn’t occur. Choose planes a and b to form a dihedral angle whose measure θ is incommensurable with 360°, and choose plane c perpendicular to a and b, so that the reflection across c commutes with the reflections across a and b. Then Φ can be reduced to $(ab)^k c$ or $(ab)^k$ for some integer k, depending on whether an odd or even number of cs occur. The former is impossible because ι is a direct motion. The latter is impossible unless $k = 0$, because $(ab)^k$ is a rotation through angle $2k\theta$ about the axis of the dihedral angle.

While the argument in the previous paragraph is simple, it’s not the last word. To match the theorem that every three-dimensional rigid motion is a composition of a sequence of at most *four* reflections across planes, *all* of a, b, c, and d must be allowed to occur in Φ. Thomsen’s equation in two dimensions seems to suggest that there is a sequence Φ that *can’t* be reduced to the identity just by repeatedly canceling aa, bb, cc, and dd. But that’s not so! Thomsen’s equation has *no* three-dimensional counterpart. This was established a generation after Thomsen’s and Kneser’s work, although some of the techniques required for the proof had been introduced by Felix Hausdorff in 1914. There’s an elegant matrix-algebraic proof with historical notes in Wagon [**12**, Theorem 5.10].

Readers comfortable with higher-dimensional versions of geometric arguments such as that in the paragraph before last can extend the three-dimensional result recursively to all higher dimensions. Another approach is found in Dekker [**5**], the source of Wagon’s argument, which uses n-dimensional matrix algebra. The result is this:

Theorem 12. *Let Φ be a finite sequence of entries selected from a list $\{a,b,c,d,\ldots\}$ of $n+1$ symbols representing hyperplanes in $\mathbb{R}^n$ with $n \ge 3$. Suppose the corresponding composition of reflections is the identity ι for every choice of the hyperplanes. Then*

Φ must include two identical entries in succession. (*Thus, Φ can be reduced to the empty sequence by removing pairs such as $aa, bb, \ldots$.*)

5. HISTORICAL NOTES. This section describes Kneser’s long and successful career and Thomsen’s unfortunate one, and provides references to further information on the geometry in this paper.

Hellmuth Kneser was born in 1898 in Dorpat (now called Tartu, in Estonia). His father Adolf Kneser, professor of mathematics at the university there, was instrumental in propagating Hilbert’s abstract approach to analysis and applying it to physical problems. In 1916, the father became professor at Breslau (now Wrocław, in Poland), and the son began study with Hilbert’s former student Erhard Schmidt. Kneser transferred to Göttingen in 1918, where he finished his doctoral studies under Hilbert, writing a dissertation on quantum theory.

Rather than specializing, Kneser embraced many areas of mathematics and became a prolific researcher, writer, and teacher. He held positions at Greifswald from 1925 until 1937 and at Tübingen during the years 1937 to 1967. Over the course of this long career he made important contributions to topology, to the theory of functions of several complex variables, to applications in social sciences, and to the study of relationships between geometry and algebra. His first doctoral student was Reinhold Baer, who later made major contributions to algebra and geometry. Kneser was an important figure in the founding and early direction of the Mathematisches Forschungsinstitut at Oberwolfach. He died in 1973.

Gerhard Thomsen was born in Hamburg in 1899, the son of a physician. He served for a year in World War I, then became one of the first students of the new university at Hamburg. Thomsen completed the Ph.D. in 1923 with a dissertation on differential geometry. He served as assistant in Karlsruhe and Hamburg, studied a year with Tullio Levi-Civita in Rome, then presented his *Habilitationsschrift* in Hamburg, on a problem in gravitational physics. In 1929 he became Ausserordentlicher Professor at Rostock. There he continued research on differential geometry, collaborating with Wilhelm Blaschke, and also published work in mathematical physics and foundations of geometry.

From the last mentioned work stems the present paper. Thomsen discovered how to formulate many familiar properties of plane figures in terms of equations involving compositions of half-turns and reflections. His 1933 book [**9**] contains that work and the start of its applications to the axiomatic foundations of geometry; reference [**10**] is a translated excerpt. Thomsen began to consider analogous three-dimensional questions. His Ph.D. student H. Boldt developed those into a full theory, published in 1934 [**2**].

This was the time of the Nazification turmoil throughout Germany. Ruth Carlsen has described in detail its effect on the Rostock university [**3**], [**4**]. In reaction, Thomsen delivered in 1933 an inflammatory lecture [**11**] that seemed to support some Nazi aims but attacked Nazi suppression of education in the sciences. That, and perhaps other concerns, evidently attracted the attention of the secret police. (A colleague asked the

Mecklenburg state attorney general, who had received a copy of the lecture, to protect Thomsen from the Gestapo [**7**].) Thomsen died, an apparent suicide, on a Rostock railroad track in 1934.

Thomsen's last work was eventually incorporated into Friedrich Bachmann's project to formulate foundational studies of many areas of geometry in terms of reflections [**1**]. The present author's interest in this subject was inspired in large part by Bachmann.

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