TAYLOR'S THEOREM

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Taylor's theorem plays a central role in numerical analysis, providing a method for approximating arbitrary functions by polynomials, and estimating the errors. There are several common proofs. The one that follows, from James WOLFE, "A proof of Taylor's formula," *American mathematical monthly*, 60 (1953), 415-416, is appropriate because it depends on an extension of Rolle's theorem, like some other results fundamental to this subject.

According to Rolle's theorem, if a function g is differentiable on an open interval $I = (a, b)$, continuous at a and b, and $g(a) = g(b) = 0$, then $g'(\xi) = 0$ for some ξ in I. The extension needed here is concerned with higher derivatives:

> if g has n continuous derivatives on the closed interval $[a, b]$, $g^{(n+1)}$ exists on its interior I, $g^{(j)}(a) = 0$ for $j = 0$ to n, and $g(b) = 0$, then $g^{(n+1)}(\xi) = 0$ for some ξ in I.

The proof is simple. First, g satisfies the hypotheses of Rolle's theorem on the interval $[a, b]$, so $g'(x_1) = 0$ for some x_1 in I. Now apply the same argument to g' on the interval $I_1 = [a, x_1]$ to determine x_2 in I_1 with $g''(x_2) = 0$. Continue the process until you've found $\xi = x_{n+1}$ in I_n with $g^{(n+1)}(x_{n+1}) = 0$.

With this preliminary step completed, you can turn to approximation by polynomials. Consider, as an example, a fourth degree polynomial and its derivatives:

$$\begin{aligned}
P(x) &= P^{(0)}(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + a_4(x-a)^4 \\
P'(x) &= P^{(1)}(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + 4a_4(x-a)^3 \\
P''(x) &= P^{(2)}(x) = 2a_2 + 2{\cdot}3a_3(x-a) + 3{\cdot}4a_4(x-a)^2 \\
&\ P^{(3)}(x) = 2{\cdot}3a_3 + 2{\cdot}3{\cdot}4a_4(x-a) \\
&\ P^{(4)}(x) = 2{\cdot}3{\cdot}4a_4
\end{aligned}$$

This should convince you that, in general, for an nth degree polynomial

$$P(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n \qquad (*)$$

the formula

$$P^{(j)}(a) = j!\,a_j$$

holds for $j = 0$ to n. You can use it in reverse: *to construct a polynomial* $(*)$ *with prescribed derivatives* $P^{(j)}(a) = p_j$, *set* $a_j = p_j/j!$ *for* $j = 0$ *to* n.

Now let f be any function with n derivatives at a point a. Use the method of the previous paragraph to construct a polynomial $T_{n,a}(x)$ with the same derivatives there:

$$T_{n,a}(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n$$

$$\left.\begin{aligned} a_j &= \frac{f^{(j)}(a)}{j!} \\ T_{n,a}^{(j)}(a) &= f^{(j)}(a) \end{aligned}\right\} \quad \text{for } j = 0 \text{ to } n.$$

$T_{n,a}$ is called the nth degree *Taylor* polynomial for f at a. When $a = 0$, it's also called a *Maclaurin* polynomial.

If f is sufficiently smooth, you can approximate values $f(x)$ by those of Taylor polynomials even at arguments x distant from a. *Taylor's theorem* lets you determine how good an approximation it is:

> Suppose f has n continuous derivatives on the closed interval $[a,b]$, and the $n+1$st exists on its interior I. Then there exists ξ in I such that
>
> $$f(b) = T_{n,a}(b) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(b-a)^{n+1}.$$

The right hand term is called the *error term*: it tells how far the Taylor polynomial deviates from the function value at x. There are several versions of Taylor's theorem with different but equivalent error terms; this one is known as *Lagrange's*. If you can estimate the size of $f^{(n+1)}(\xi)$, then you can estimate the accuracy of the Taylor approximation. Note that Taylor's theorem for $n = 0$ is simply the mean-value theorem.

To follow Wolfe's proof, define

$$k = \frac{f(b) - T_{n,a}(b)}{(b-a)^{n+1}}$$

$$g(x) = f(x) - T_{n,a}(x) - k(x-a)^{n+1}.$$

for all x. Then for $j = 0$ to n,

$$g^{(j)}(a) = f^{(j)}(a) - T_{n,a}^{(j)}(a) - k\left.\frac{d^j}{dx^j}(x-a)^{n+1}\right|_{x=a}$$

$$= f^{(j)}(a) - f^{(j)}(a) - k \cdot 0 = 0.$$

Moreover, $g(b) = f(b) - T_{n,a}(b) - k(b-a)^{n+1} = 0$ by the definition of k. By the extended Rolle's theorem, $g^{(n+1)}(\xi) = 0$ for some ξ in I. That means

$$0 = g^{(n+1)}(\xi)$$
$$f^{(n+1)}(\xi) - T_{n,a}^{(n+1)}(x) - k\frac{d^{n+1}}{dx^{n+1}}(x-a)^{n+1}\Bigg|_{x=a}$$
$$= f^{(n+1)}(\xi) - (n+1)!\,k$$
$$0 = f^{(n+1)}(\xi) - (n+1)!\frac{f(b) - T_{n,a}(b)}{(b-a)^{n+1}}.$$

The equation in Taylor's theorem is just a rearrangement of this last one.

As an example, you can construct some Maclaurin polynomials for $f(x) = \sin x$ at $a = 0$. Note that

$$f(0) = 0 \qquad f'(0) = 1 \qquad f''(0) = 0 \qquad f^{(3)}(0) = -1$$

and subsequent derivatives repeat these values cyclically. Each polynomial of odd degree n is identical to the next one, hence its error term has degree $n + 2$. Since the derivatives of the sine are positive or negative sines or cosines, you can easily find error bounds. For $n = 7$, for instance, here's the Maclaurin polynomial, its error term, and an error bound:

$$T_{7,0}(x) = T_{8,0}(x) = x - x^3/3! + x^5/5! - x^7/7!$$
$$\frac{\left|f^{(9)}(\xi)\right|}{9!}|x|^9 \le \frac{|x|^9}{9!}.$$

The following table shows the values of these bounds for $x = 3\pi/2$ and $n = 1$ to 13. The polynomial approximations to $\sin x = -1$ are also shown, as well as the actual error.

m	Error Bound	$T_{m,0}(x)$	Actual Error
1	1.7E+1	4.7124	5.7E+0
3	1.9E+1	−12.7286	−1.2E+1
5	1.0E+1	6.6367	7.6E+0
7	3.2E+0	−3.6023	−2.6E+0
9	6.4E−1	−0.4444	5.6E−1
11	9.1E−2	−1.0819	−8.2E−2
13	9.6E−3	−0.9911	8.8E−3

Figure 1 compares graphs of all but the last of these polynomials with that of the sine.

Because of the size of the errors, Maclaurin polynomials do not provide a reasonable way of calculating $\sin x$ except for very small values of x. Other methods are used in practice.

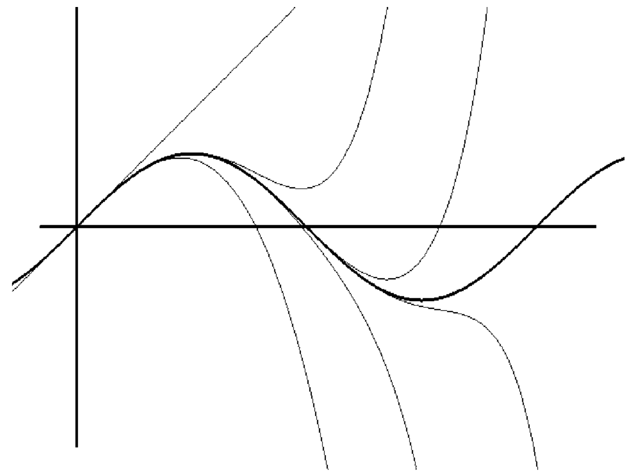

Figure 1 $\sin x$ and its first few Maclaurin polynomials