SQUARE ROOT BY HAND

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To compute the square root of a number x, group its digits into pairs proceeeding left and right from the point. Find the largest digit q with $q^2 \leq$ the leftmost pair. Write it on top, as the first digit of $\sqrt{x}$. Subtract q^2 from the first pair. Obtain subsequent digits of $\sqrt{x}$ by repeating the following steps (1) and (2). Stop if the remainder becomes zero.

(1) Append to the remainder the next pair; call the resulting number e. Double the part of the root already found, append a zero, call the resulting number d, and write it left of e.

(2) How many times does d go into e? Write the quotient q as the next digit of $\sqrt{x}$, change d to $d + q$, compute $p = q(d + q)$, and write p under e. If $p > e$, repeat the instructions of the last sentence with $q - 1$ in place of q. Subtract the final p from e.

Place the point in $\sqrt{x}$ above the one in x. An example is carried out to seven digits accuracy at right. It would have terminated with an exact root had the problem been to find $\sqrt{650.25}$. This problem was chosen, however, to illustrate how to handle more complicated situations, particularly when $q = 0$.

```
         |06 50.25 71 30 99 72

        q↘
          2
         |06 50.25 71 30 99 72
         |04
          2

          2
         |06 50.25 71 30 99 72
         |04
d = 40   | 2 50 = e

            q↘
              5
          2   6
         |06 50.25 71 30 99 72
         |04
d = 40   | 2 50 = e
     6   | 2 76
     5   | 2 25
            25

         2  5. 5  0  0  1  3
        |06 50.25 71 30 99 72
        |04
  4ϕ    | 2 50 = e
  5     | 2 25
   50 ϕ |  25 25
      5 |  25 25
       510ϕ |  0 71
          0 |    0
        5100ϕ |  71 30
            0 |      0
        51000ϕ |  71 30 99
             1 |  51 00 01
       510002ϕ |  20 30 98 72
             3 |  15 30 00 69
```

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Here's a justification of the method. First, fill in the "missing" scratchwork digits and points as shown. Denote by $a_1, a_2, \ldots$ successive approximations to $\sqrt{x}$ obtained from the digits q, and denote by $r_1, r_2, \ldots$ the corresponding remainders occurring in the calculation. You can show that

$$(*) \qquad r_k = x - a_k^2$$

for every k. For $k = 1$, this is true by the definition of the first digit q of $\sqrt{x}$.

	2 5.
	06 50.25 71 30 99 72
	04 00.00 00 00 00 00
4ϕ	02 50.25 71 30 99 72
5	02 25.00 00 00 00 00
50ϕ	00 25.25 71 30 99 72
$a_1 =$	20.0
$r_1 =$	250.2571309972
$a_2 =$	25.0
$r_2 =$	25.2571309972

This just establishes (*) for one k value, but now you can show that once it holds for one value it must also hold for the next. In fact, the next step (1) determines a digit q such that $a_{k+1} = a_k + c$, where $c = q \cdot 10^p$ for some p. Following the calculation of d, e, and p in steps (1) and (2), you can see that

$$r_{k+1} = r_k - c(2a_k + c) = x - a_k^2 - 2a_k c - c^2 = x - (a_k + c)^2 = x^2 - a_{k+1}^2.$$

This shows that (*) holds for the value $k + 1$. Your argument is completed by considering (*) for $k = 1$, then stepping to the next k, then the next, and so on: (*) must hold for *every* k.

Exercises

1. Compute some square roots by hand to the same number of digits provided by your calculator. Do the results agree?

2. Compute a fourth root $x^{1/4}$ to this number of digits by finding $a = \sqrt{x}$ to more than twice as many digits, then computing $\sqrt{a} = x^{1/4}$. Does the result agree with your calculator?

3. There's a somewhat similar hand method for computing cube roots. Look it up, do some examples, and explain it.

4. Find out the history of these square and cube root methods.