

RIEMANN'S INTEGRAL DEFINITION

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Riemann published his definition of the definite integral in the paper „Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe“, *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, XIII (1854). In English, the title reads “On the representation of a function by a trigonometric series.” I’ve reproduced the critical paragraphs on Page 2. They read:

The vagueness that persists in some fundamental points of the theory of definite integrals requires us to put down something about the definite integral concept and the scope of its validity.

Thus, first: what do we understand by $\int_a^b f(x)dx$?

To make this secure, we take a sequence of values x_1, \dots, x_n between a and b and denote the short distances $x_1 - a$ by δ_1 , $x_2 - x_1$ by δ_2 , ..., $b - x_{n-1}$ by δ_n . [For each k , ε_k] denotes a positive number less than 1. Then the sum

$$S = \delta_1 f(a + \varepsilon_1 \delta_1) + \delta_2 f(x_1 + \varepsilon_2 \delta_2) + \delta_3 f(x_2 + \varepsilon_3 \delta_3) + \dots + \delta_n f(x_{n-1} + \varepsilon_n \delta_n)$$

depends on the choice of the intervals δ and the quantities ε . Now if the sum has the property that however the δ and ε values are chosen, as long as the δ all become arbitrarily small together, it approaches a fixed limit A , then this value is called $\int_a^b f(x)dx \dots$

If the sum doesn't have this property, then $\int_a^b f(x)dx$ has no meaning.

The main purpose of Riemann's paper was to determine the extent to which the Fourier series techniques of advanced calculus apply to various problems in mathematical physics. He needed to compute integrals of the functions useful in that field. The theory of integrals of *continuous* functions had been worked out before 1854—notably by Cauchy. But Riemann needed integrals of some *discontinuous* functions, and complained that the familiar theory was vague on those matters. He had to fix that before he could proceed with the Fourier analysis.