QUADRATIC FORMULA

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You've met the *quadratic formula* in algebra courses. The solution of the quadratic equation

$$ax^2 + bx + c = 0$$

with specified real number coefficients $a \neq 0$, b, and c is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

You can derive the formula as follows. First, divide the quadratic by a to get the equivalent equation

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

Now substitute $x = y + d$. You'll choose d later so that the resulting equation is easy to solve. Making the substitution, you get

$$(y+d)^2 + \frac{b}{a}(y+d) + \frac{c}{a} = 0.$$

Work this out, ignoring some details that won't be necessary:

$$y^2 + 2dy + d^2 + \frac{b}{a}y + \text{constants} = 0$$
$$y^2 + \left(2d + \frac{b}{a}\right)y + \text{constants} = 0.$$

if the y coefficient were zero, then you could move the constants to the other side and solve for y by taking the square root. Thus you can find y easily if you let

$$d = -\frac{b}{2a}.$$

Do that and work out the details. The last equation displayed becomes

$$y^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$y^2 = \frac{b^2 - 4ac}{4a^2}.$$

The quantity $D = b^2 - 4ac$ is called the *discriminant* of the quadratic, and

$$y = \pm\frac{\sqrt{D}}{2a}.$$

Finally, the desired solution is

$$x = d + y = -\frac{b}{2a} \pm \frac{\sqrt{D}}{2a}$$

—the quadratic formula. You can determine the *number* of different roots by inspecting the discriminant:

$$D = b^2 - 4ac \begin{cases} < 0 & \textit{no} \quad \text{root} \\ = 0 & \textit{one} \quad \text{root} \\ > 0 & \textit{two} \quad \text{roots.} \end{cases}$$

Examples

$x^2 = 0$
$a = 1,\ b = c = 0$
$D = 0$: one root
$$x = \frac{-0 \pm \sqrt{0}}{2} = 0$$

$x^2 = 1$
$a = 1,\ b = 0,\ c = -1$
$D = 4 > 0$: two roots
$$x = \frac{-0 \pm \sqrt{4}}{2} = \pm 1$$

$x^2 = -1$
$a = 1,\ b = 0,\ c = 1$
$D = -4 < 0$: no root

$x^2 - 3x + 2 = 0$
$a = 1,\ b = -3,\ c = 2$
$D = 1 > 0$: two roots
$$x = \frac{3 \pm \sqrt{1}}{2} = 2 \text{ or } 1$$

$3x^2 - 4x - 5 = 0$
$a = 3,\ b = -1,\ c = -2$
$D = 76 > 0$: two roots
$$x = \frac{4 \pm \sqrt{76}}{6} = \frac{2 \pm \sqrt{19}}{3}$$
$$\approx 1.15 \text{ or } -0.786$$

Exercises

1. Solve these equations: $x^2 + x + 1 = 0$, $x^2 + x - 1 = 0$, $x^2 - x + 1 = 0$, $x^2 - x - 1 = 0$.

2. Solve these: $x^4 - 5x^2 + 6 = 0$ (find x^2 first), $4/x + x = 5$, $\sqrt{x} + 2 = 3x$ (check your answer).

3. Show that when $D \geq 0$, $-b/a$ is the sum of the roots of $ax^2 + bx + c = 0$, and c/a is their product.

4. Investigate the history of this method.

Solutions

1. Solve these equations:

$x^2 + x + 1 = 0$ $D = -3$: no root
$x^2 + x - 1 = 0$ $D = 5$: two roots $x = (-1 \pm \sqrt{5})/2 \approx -1.62$ or 0.618
$x^2 - x + 1 = 0$ $D = -3$: no root
$x^2 - x - 1 = 0$ $D = 5$: two roots $x = (1 \pm \sqrt{5})/2 \approx 1.62$ or -0.618

2. Solve these: $x^4 - 5x^2 + 6 = 0$ (find x^2 first),

$(x^2)^2 - 5x^2 + 6 = 0$, hence
$x^2 = (5 \pm \sqrt{1})/2 = 3$ or 2 by the quadratic formula, so
$x = \pm\sqrt{3}$ or $\pm\sqrt{2}$;

$4/x + x = 5$,

$4 + x^2 = 5x$, hence
$x^2 - 5x + 4 = 0$, hence
$x = 4$ or 1 by the quadratic formula;

$\sqrt{x} + 2 = 3x$ (check your answer),

$\sqrt{x} + 2 = 3(\sqrt{x})^2$, hence
$3(\sqrt{x})^2 - \sqrt{x} + 2 = 0$, hence
$\sqrt{x} = -2/3$ or 1 by the quadratic formula, but
the first alternative is impossible because $\sqrt{x} \geq 0$, hence
$\sqrt{x} = 1$, i.e. $x = 1$.

3. Show that when $D \geq 0$, $-b/a$ is the sum of the roots of $ax^2 + bx + c = 0$, and c/a is their product.

$(-b + \sqrt{D})/(2a) + (b - \sqrt{D})/(2a) = (-2b)/(2a) = -b/a$;
$(-b + \sqrt{D})/(2a) \cdot (-b - \sqrt{D})/(2a) = [(-b)^2 - (\sqrt{D})^2]/(4a^2) = [b^2 - (b^2 - 4ac)]/(4a^2) =$
$4ac/(4a^2) = c/a$.