FIRST ORDER ORDINARY DIFFERENTIAL EQUATION INITIAL VALUE PROBLEMS

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To pose a *first-order ordinary differential equation initial value problem*, specify

- a nonempty closed interval $I = [x_0, x_0 + H]$,
- an initial value u_0, and
- a function $f(x,y)$ defined for all x in I and for all y.

A *solution* of the problem is a differentiable function u on I such that

- $u(x_0) = u_0$, and
- $u'(x) = f(x, u(x))$ for all x in I.

The problem is called *well posed* if there's a unique solution that depends continuously on the data x_0, H, u_0, and f. (This notion of continuity has to be defined precisely.) Various conditions on the data are known which guarantee that the problem is well posed. Here's an example.

Theorem 1. The problem is well posed if f is continuous and there exists L such that for all x in I and all y and z, $|f(x,y) - f(x,z)| \le L|y - z|$.

The proof of theorem 1, due to Émile Picard, is too long to give here—see the references by Pennisi and Shampine & Gordon. L is a Lipschitz constant; this suggests that you might use a fixpoint method to prove the theorem. That's indeed possible. The ordinary differential equation initial value problem is equivalent to the following integral equation problem: find a function u such that for all x,

$$u(x_0) = u_0 + \int_{x_0}^{x} f\big(t, u(t)\big)\,dt. \qquad (*)$$

The right hand side of equation (*) defines an operator Ω on the space $\mathcal{C}$ of continuous functions u on I:

$$\Omega(u)(x) = u_0 + \int_{x_0}^{x} f\big(t, u(t)\big)\,dt.$$

The solution u is a fixpoint of Ω: u satisfies equation (*) just when $\Omega(u) = u$. You can use the constant L to determine a Lipschitz constant for Ω (with respect to a norm $\| \ \|$ on the space $\mathcal{C}$) as follows:

$$\left|\Omega(u)(x) - \Omega(u)(y)\right| = \left|\int_{x_0}^{x} \left[f\left(t,u(t)\right) - f\left(t,v(t)\right)\right]dt\right|$$

$$\le \int_{x_0}^{x} \left|f\left(t,u(t)\right) - f\left(t,v(t)\right)\right|dt \le L\int_{x_0}^{x} \left|u(t) - v(t)\right|dt$$

$$\left\|\Omega(u) - \Omega(v)\right\| = \max_{x\in I}\left|\Omega(u)(x) - \Omega(v)(x)\right|$$

$$\le L\int_{x_0}^{x_0+H} \left|u(t) - v(t)\right|dt \le LH \max_{x\in I}\left|u(x) - v(x)\right| = LH\left\|u - v\right\|.$$

You can use general topology techniques to establish notions of convergence and continuity for the space $\mathcal{C}$ with respect to this norm. With them, you can fit the usual fixpoint ideas into this new context.

Properly formulated problems arising in applications are ordinarily well posed. However, some cases may be borderline, especially if the continuity condition on f is relaxed. In borderline cases, some simplifications and round-off errors may destroy well-posedness. Thus a knowledge of theorems about this concept, though not often critical in applications, is sometimes necessary for analyzing unexpected results in the solution process.

Here's a crude graphical method for solving an ordinary differential equation initial value problem: set $y_0 = u_0$, plot the point $<x_0, y_0>$, compute $y_0' = f(x_0, y_0)$, draw a line segment with slope y_0' to a point $<x_1, y_1>$, compute $y_1' = f(x_1, y_1)$, and continue until $x_n \ge x_0 + H$. The segments form the graph of an approximation $y(x)$ to the true solution $u(x)$, as shown in figure 1. Formalized as an algorithm, this is known as *Euler's method*:

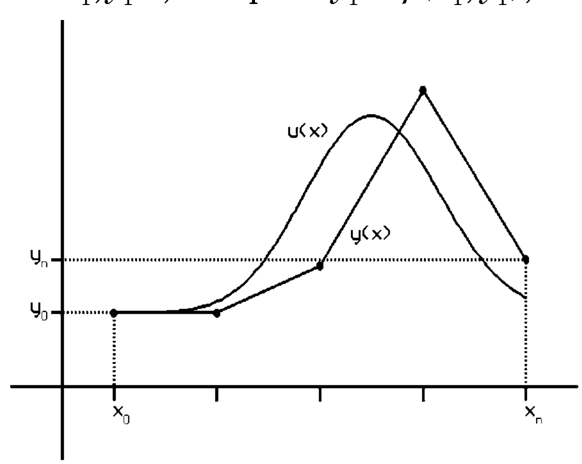

Figure 1

given

- x_0 : the initial argument
- u_0 : the initial value
- H : the interval length
- N : the number of steps
- $f(x, y)$: the function

compute

$h = H/N$: the step size
$y_0 = u_0$, and

for $n = 0$ to $N - 1$
$y_{n+1} = y_n + hf(x_n, y_n)$
$x_{n+1} = x_n + h$.

In this version of the algorithm the step size h is constant. Some sophisticated implementations determine when the function f is nice enough that the current h permits a sufficiently accurate solution, bad enough that smaller steps are necessary; or so smooth that larger steps would be faster but just as accurate. Then they vary the step size accordingly.

Here's an example problem: given

$x_0 = 0, \ u_0 = 1, \ H = 1,$
$N,$
$f(x, y) = y,$

find a function u on $I = [x_0, x_0 + H] = [0, 1]$ such that $u(x_0) = u_0 = 1$ and $u'(x) = f(x, u(x)) = u(x)$ for all x in I. The familiar solution is the exponential function. Euler's method yields $h = 1/N$, $y_0 = 1$, and for $n = 0$ to $N - 1$,

$$y_{n+1} = y_n + hy_n = (1 + h)y_n$$
$$x_{n+1} = x_n + h.$$

That is,

$$y_n = (1 + h)^n$$
$$x_n = nh.$$

Since $1 = x_N = Nh$,

$$\lim_{N \to \infty} y_N = \lim_{N \to \infty} \left(1 + \frac{1}{N}\right)^N.$$

This last limit is just e, hence

$$\lim_{N \to \infty} y_N = e = u(1).$$

Thus, Euler's method with N steps produces an approximation y_N that converges to the true solution at the end of the interval, as N increases.

You need a close analysis to study convergence of Euler's method in general. Define the error at the nth step to be $\varepsilon_n = y_n - u_n$, where $u_n = u(x_n)$. A major step in finding a bound for $|\varepsilon_n|$ is solving a common recursive inequality, using the following result.

Theorem 2. Suppose A and B are any real numbers and $\varepsilon_0, \varepsilon_1, \ldots$ is a sequence of real numbers such that $\varepsilon_{n+1} \le A\varepsilon_n + B$ for $n = 0, 1, \ldots$. Then

$$\varepsilon_n \le A^n \varepsilon_0 + \frac{A^n - 1}{A - 1} B$$

for all n. Moreover, if $\delta > 0$ and $A = 1 + \delta$, then

$$\varepsilon_n \le e^{n\delta} \varepsilon_0 + \frac{e^{n\delta} - 1}{\delta} B.$$

Proof.
$$\begin{aligned} \varepsilon_1 &\le A\varepsilon_0 + B \\ \varepsilon_2 &\le A\varepsilon_1 + B \le A^2\varepsilon_0 + AB + B \\ \varepsilon_3 &\le A\varepsilon_2 + B \le A^3\varepsilon_0 + A^2B + AB + B \\ &\vdots \\ \varepsilon_n &\le A^n\varepsilon_0 + \sum_{k=0}^{n-1} A^k B = A^n\varepsilon_0 + \frac{A^n - 1}{A - 1} B. \end{aligned}$$

If $\delta > 0$ and $A = 1 + \delta$, then $A \le e^\delta$, so $A^n \le e^{n\delta}$. ♦

Now you can perform the error analysis of Euler's method for solving the initial value problem $u' = f(x,u)$, $u(x_0) = u_0$ for x in the interval $I = [x_0, x_0 + H]$. Suppose f is bounded and has bounded partial derivatives. Then $L = \max|\partial f/\partial u|$ is a Lipschitz constant:

$$|f(x,y) - f(x,z)| = \left| (y - z) \frac{\partial f}{\partial u}\bigg|_{x=\eta} \right| \le L|y - z|$$

for some η between y and z by the mean-value theorem. Also, u'' is bounded:

$$u''(x) = \frac{\partial f}{\partial x}\bigg|_{x,u(x)} + \frac{\partial f}{\partial u}\bigg|_{x,u(x)} u'(x) = \frac{\partial f}{\partial x}\bigg|_{x,u(x)} + \frac{\partial f}{\partial u}\bigg|_{x,u(x)} f\big(x,u(x)\big).$$

Let $M = \max|u''(x)|$. Define $x_1, x_2, \ldots, x_N$ and H as before, so that $Nh = H$, and compare the solution values $u_n = u(x_n)$ with the Euler approximations, using Taylor's theorem:

$$\begin{aligned} y_{n+1} &= y_n + hf(x_n, y_n) \\ u_{n+1} &= u_n + hu'(x_n) + \tfrac{1}{2}h^2u''(\xi_n) \\ &= u_n + hf(x_n, u_n) + \tfrac{1}{2}h^2u''(\xi_n) \end{aligned}$$

for some ξ_n between x_n and x_{n+1}. Compute

$$\begin{aligned} |\varepsilon_{n+1}| &= |y_{n+1} - u_{n+1}| \\ &= |[y_n - u_n] + H[f(x_n, y_n) - f(x_n, u_n)] - \tfrac{1}{2}h^2 u''(\xi_n)| \\ &= \left| \left(1 + h \frac{\partial f}{\partial u}\bigg|_{x_n, \eta_n} \right)(y_n - u_n) - \frac{1}{2}u''(\xi_n) \right| \\ &\le (1 + hL)|y_n - u_n| + \tfrac{1}{2}h^2M \\ &= (1 + hL)\varepsilon_n + \tfrac{1}{2}h^2M. \end{aligned}$$

Thus $|\varepsilon_{n+1}| \le A\varepsilon_n + B$ where $A = 1 + hL$ and $B = \tfrac{1}{2}h^2M$. By theorem 2, since $\varepsilon_0 = 0$, this inequality implies

$$\varepsilon_N \le \frac{e^{NhL} - 1}{hL}\frac{h^2M}{2} = \frac{hM}{2L}\left(e^{HL} - 1\right)$$

$$\lim_{N\to\infty} \varepsilon_N = 0.$$

That is, the approximation y_n at the last step of the interval approaches the true solution u_n as N increases. Because ε_N is bounded by a constant times the first power of H, Euler's is called a *first-order* method.

Euler's is merely the simplest of a family of algorithms, called *single-step* methods. Using appropriate definitions suggested by Henrici (see the references), you can extend the error analysis to apply to all these methods. A single-step method is an algorithm of the following type:

given x_0, u_0, H, and N as before, and
a function $\Phi(x, y, h)$ defined for all x, y, and H under consideration,

compute $h = H/N$ and $y_0 = u_0$ as before, and
for $n = 0$ to $N - 1$
$y_{n+1} = y_n + h\Phi(x_n, y_n, h)$
$x_{n+1} = x_n + H.$

For Euler's method, $\Phi(x,y,h) = f(x,y)$. Note that in general, y_{n+1} depends only on y_n, not on values y_k for $k < n$. That's why these are called single-step methods. You can regard single-step algorithms as *difference* equation approximations

$$\frac{y_{n+1} - y_n}{h} = \Phi(x_n, y_n, h)$$

to the differential equation

$$\lim_{h\to 0} \frac{u(x+h) - u(x)}{h} = f\left(x, u(x)\right).$$

The solution $y_0, y_1, \ldots$ of the difference equation approximates the sequence of solution values $u_0, u_1, \ldots$ of the differential equation; the latter sequence is not normally itself a solution of the difference equation. One of the factors in the error analysis is the amount by which the sequence $u_0, u_1, \ldots$ fails to satisfy the difference equation: the *local truncation error*

$$t_n = \frac{u_{n+1} - u_n}{h} - \Phi(x_n, u_n, h).$$

(Some authors define this concept slightly differently.)

Theorem 3. If $f(x,u)$ is bounded and has bounded partial derivatives, and t_n is the local truncation error at the nth step of Euler's method, then $|u''|$ is bounded and $|t_n| \le \tfrac{1}{2} h \max |u''|$.

Proof. An earlier calculation showed that $|u''|$ is bounded. Now use Taylor's theorem:

$$t_n = \frac{u_{n+1} - u_n}{h} - f(x_n, u_n) = \frac{hu'(x_n) + \tfrac{1}{2}u''(\xi)}{h} - f(x_n, u_n) = \tfrac{1}{2} h u''(\xi). \blacklozenge$$

The next result contains the error analysis for general single step methods.

Theorem 4 Consider a single step method $y_{n+1} = y_n + h\Phi(x_n, y_n, h)$ where the function Φ satisfies a Lipschitz condition $|\Phi(x,y,h) - \Phi(x,z,h)| \le L|y - z|$. If the local truncation error satisfies an equality $|t_n| \le h^p M$ for some constant M, then the approximation y_n at the last step of the interval approaches the true solution u_n as N

increases. Moreover, the error at that step is bounded by a constant times h^p: the method has order p.

Proof. As in the analysis of Euler's method, compare the solution values $u_n = u(x_n)$ with their approximations, but use the condition on the local truncation error instead of applying Taylor's Theorem directly:

$$y_{n+1} = y_n + h\Phi(x_n, y_n, h)$$

$$\begin{aligned}\varepsilon_{n+1} &= y_{n+1} - u_{n+1} \\ &= y_n - u_n + h\Phi(x_n, y_n, h) - (u_{n+1} - u_n) \\ &= \varepsilon_n + h\Phi(x_n, y_n, h) - h\Phi(x_n, u_n, h) + h\Phi(x_n, u_n, h) - (u_{n+1} - u_n) \\ &= \varepsilon_n + h\Phi(x_n, y_n, h) - h\Phi(x_n, u_n, h) - h t_n\end{aligned}$$

$$|\varepsilon_{n+1}| \le |\varepsilon_n| + hL|y_n - u_n| + h^{p+1}M = (1 + hL)|\varepsilon_n| + h^{p+1}M.$$

By theorem 2, this last inequality implies

$$\varepsilon_n \le \frac{e^{NhL} - 1}{hL} h^{p+1} M = \frac{h^p M}{L}\left(e^{HL} - 1\right). \blacklozenge$$

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