MEAN-VALUE THEOREMS OF DIFFERENTIAL CALCULUS

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This note describes three theoretical results used in several areas of differential calculus, and a related concept, *Lipschitz constants*. Let $a < b$ and I be the closed interval $[a,b]$.

Rolle's theorem. Suppose a function f is defined and continuous at each point of I, and differentiable at each interior point. If $f(a) = f(b)$, then $f'(x_0) = 0$ for some interior point x_0.

Proof. If f is constant, then $f'(x_0) = 0$ for *all* x_0 in I. Otherwise, f has an extreme value at an interior point x_0, and

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} .$$

The quotient changes sign at $x = x_0$, so its limit must be 0. ♦

Cauchy's mean-value theorem. Suppose functions f and g are defined and continuous at each point of I, and differentiable at each interior point. Then

$$[f(b) - f(a)]\, g'(x_0) = [g(b) - g(a)]\, f'(x_0)$$

for some interior point x_0.

Proof. Define $F(x) = [f(b) - f(a)]\, g(x) - [g(b) - g(a)]\, f(x)$ for all x in I, so that $F(a) = F(b)$. By Rolle's Theorem, $F'(x_0) = 0$ for some interior point x_0. ♦

Standard mean-value theorem. Suppose a function f is defined and continuous at each point of I, and differentiable at each interior point. Then

$$f(b) - f(a) = f'(x_0)(b - a)$$

for some interior point x_0.

Proof. Define $g(x) = x$ for all x in I and apply Cauchy's mean-value theorem. ♦

Lipschitz constants. The mean-value theorem is frequently cited to show the existence of certain numbers used in computations involving functions f defined on a set I. A *Lipschitz constant* for f on I is a number L such that

$$|f(x) - f(x')| \le L\ |x - x'|$$

for all x and x' in I. If I is a finite closed interval $[a,b]$, f is continuous on I, $f'(x_0)$ exists at each interior point x_0 of I, and $|f'(x_0)| \le L$ for each such x_0, then L is a Lipschitz constant, because

$$|f(x) - f(x')| = |f'(x_0)(x - x')| = |f'(x_0)|\,|x - x'| \le L\,|x - x'|$$

for some interior point x_0, by the mean-value theorem.