Mashers mathematical

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In August, 1997, shortly after the MAA MathFest in Atlanta, Georgia, at a nearby antique store, a mathematician caused a disturbance. "Look: a *masher*! And there's another! They're different! And yet another!" Customers looked on in mild amazement as he rearranged the kitchenware display. "See, this masher looks like a sine wave. It's very symmetric. And that one's symmetry is broken: Its waves tumble as though they're breaking on the beach. A third has waves doubled up. And there's another, based on a rectangular grid."

For some years, the first author has been studying and collecting examples of symmetry in tool design. Kitchenware provides prime specimens. The symmetries of the masher bases in figure 1—from left to right—form finite groups commonly labeled $\mathscr{D}_2$, $\mathscr{D}_1$, and $\mathscr{C}_1$. The *Vierergruppe* $\mathscr{D}_2$ contains the identity, two half-turns, and the reflections across two perpendicular axes; $\mathscr{D}_1$ contains only the identity and one of these reflections; the trivial group $\mathscr{C}_1$ contains just the identity. Now imagine the bases extended indefinitely in two opposite directions. You see friezes similar to $\cdots$ HHHH $\cdots$, $\cdots$ MWMW $\cdots$, and $\cdots$ NNNN $\cdots$ whose symmetries form groups with crystallographic labels *pmm2*, *pma2*, and *p112*. If you straightened and extended a twisted-wire portion of the left masher, its symmetries would constitute an infinite cyclic group generated by a screw motion. Other mashers in the authors' employ exhibit symmetries beyond those in figure 1. For example, the symmetries of mashers based on rectangular or rhombic grids, extended in two dimensions, form wallpaper groups.[1]

Soon after the authors' first kitchenware conversation, Wheeler approached Smith with a twinkle in his eye. "I've designed the ultimate potato-rendering tool—a *Hilbert* masher!" He exhibited his draft of figure 2.[2] What's the symmetry group of its base? Do you think this has a chance as a patent proposal for kitchenware?

The closed Hilbert curve that forms the base of the ultimate masher is produced by an algorithm with a parameter, the number n of "subdivisions" of the encompassing square. Given any prescribed distance ε, however small, you can choose n large enough to ensure that every point of the square lies within distance ε of some point on the curve. Moreover, if you journey around the curve at a constant speed, you'll find that most pairs of points on the curve that are close together in the square are also close together in travel time.[3] These properties make the Hilbert curve an ideal tool for image processing. Riemersma describes its use in dithering.[4] He superimposes the curve on a gray-scale or color image, so that a journey around it provides a uniformly dense sample of the image pixels. The goal is to assign these pixels colors from a small pallette, so that
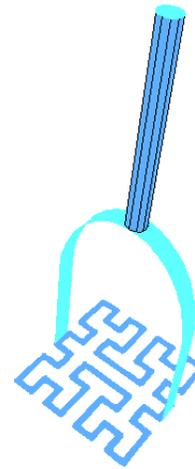
Figure 1
Antique mashers

Figure 2
Hilbert masher

the image file is as small as possible, without losing too much perceived color or gray-scale information. The color assigned to a sample pixel is computed via a formula that involves weighted differences between input and output colors of several preceding sample pixels. The structure of the Hilbert curve ensures that this smoothing process is only applied to neighboring samples, and is in fact applied to *most* neighboring samples. Processing images for publication is a complicated process—some algorithms are proprietary, and others merely difficult to identify. Perhaps a technique such as Riemersma dithering played a role in rendering figure 1!

Why did the designers of the mashers in figure 1 employ those symmetries? Are they purely decorative? Or do they enhance the tools' usefulness? Investigate some antique stores. How many symmetry groups do you find among the mashers? Analyze other tools the same way. You'll learn about symmetry in design. And watch the proprietors' reactions when you reveal that you're studying *geometry*!

Notes

1. Figure 1 and some of the text of this article are reproduced with the publisher's permission from the first author's book, *Methods of geometry*, Wiley, 2000. Consult that reference, or George E. Martin, *Transformation geometry*: *An introduction to symmetry*, Springer, 1982, for descriptions and classification of frieze and other symmetry groups.

2. Drawn with *X*(*Plore*) software, published by David B. Meredith, Prentice-Hall, 1993.

3. David Hilbert introduced this curve in the paper, "Über die stetige Abbildung einer Linie auf ein Flächenstück," *Mathematische Annalen* 38:1891, 459-460. For a precise definition of the curve and a discussion of its properties see Hans Sagan, *Space-filling curves*, New York: Springer-Verlag, 1994, section 1.1.

4. Thiadmer Riemersma, "A balanced dithering technique," *C/C++ user's journal*, December 1998, 51-58.