$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

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Define

$$g(x) = \int_0^x e^{-t^2}\,dt$$

for all real x. If $1 \le t \le x$, then $t \le t^2$, so $e^{-t^2} \le e^{-t}$ and

$$0 \le \int_1^x e^{-t^2}\,dt \le \int_1^x e^{-t}\,dt = -e^{-t}\Big|_{t=1}^{x} = -e^{-x} + e^{-1}.$$

Since $e^{-x} \to 0$ as $x \to \infty$,

$$J = \int_0^\infty e^{-t^2}\,dt = \lim_{x\to\infty} g(x) = g(1) + \lim_{x\to\infty}\int_1^x e^{-t^2}\,dt$$

exists. By the fundamental theorem of calculus and the chain rule,

$$\begin{aligned} g'(t) &= e^{-t^2} \\ \frac{d}{dx}\, g\left(\sqrt{x}\right)^2 &= 2g\left(\sqrt{x}\right)g'\left(\sqrt{x}\right)\frac{1}{2\sqrt{x}} = \frac{e^{-x}}{\sqrt{x}}\, g\left(\sqrt{x}\right) \\ &= \frac{e^{-x}}{\sqrt{x}}\int_0^{\sqrt{x}} e^{-t^2}\,dt = e^{-x}\int_0^1 e^{-xu^2}\,du \end{aligned}$$

provided $x > 0$, where $t = u\sqrt{x}$.

Now consider the function

$$f(x) = \int_0^1 \frac{e^{-x\left(1+u^2\right)}}{1+u^2}\,du$$

By a standard calculus theorem about single integrals of multivariate functions,

$$f'(x) = \int_0^1 \frac{-\left(1+u^2\right)e^{-x\left(1+u^2\right)}}{1+u^2}\,du = -\int_0^1 e^{-x-xu^2}\,du = -e^{-x}\int_0^1 e^{-xu^2}\,du$$

for all x. Moreover,

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$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

$$f(0) = \int_0^1 \frac{1}{1+u^2} du = \arctan 1 - \arctan 0 = \frac{\pi}{4},$$

f is continuous, and

$$\lim_{x\to\infty} f(x) = \int_0^1 \lim_{x\to\infty} \frac{e^{-x(1+t^2)}}{1+t^2} dt = \int_0^1 0\, dt = 0.$$

Since

$$f'(x) = -\frac{d}{dx} g\left(\sqrt{x}\right)^2,$$

$f(x) + g\left(\sqrt{x}\right)^2$ is constant for all $x > 0$. Since g is continuous too, that constant is

$$\lim_{x\to 0}\left(f(x) + g\left(\sqrt{x}\right)^2\right) = f(0) + g(0)^2 = \frac{\pi}{4}.$$

It follows that

$$\frac{\pi}{4} = \lim_{x\to\infty}\left(f(x) + g\left(\sqrt{x}\right)^2\right) = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g\left(\sqrt{x}\right)^2 = 0 + J^2.$$

Thus

$$J = \frac{\sqrt{\pi}}{2}.$$

References

Robert WEINSTOCK, "Elementary evaluations of $\int_0^\infty e^{-x^2} dx$, $\int_0^\infty \cos x^2\, dx$, and $\int_0^\infty \sin x^2\, dx$," *American mathematical monthly*, XCVII (1990), 39-42. See also the literature cited there. *Click here* to see the article.

Constantine GEORGAKIS, "A note on the Gaussian integral," *Mathematics magazine*, LXVII (1994), 47. An alternate approach, easier to motivate but harder to justify.