FUNDAMENTAL THEOREM OF ALGEBRA

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According to Gauss's *fundamental theorem of algebra*, every nonconstant polynomial p with complex coefficients has a complex root. You meet this result first in precalculus mathematics, then find that *many* studies in pure and applied mathematics are based on it. Its proof is generally postponed until graduate courses. The first proof I learned follows from Cauchy's integral theorem in complex analysis. It's elegant, but it uses a wide range of facts about continuity and differentiability of complex functions *and* the complicated apparatus of contour integrals. For years I hunted for a proof that's simple enough to present as a sidebar in undergraduate courses that *use* the fundamental theorem. I think I found one, in Remmert 1993. That author attributes the following proof to J. Argand, 1814.

By Cauchy's bound, all roots of a polynomial $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ lie in the closed disk D centered at 0 with radius

$$1 + \max_{k=0}^{n} \left| \frac{a_k}{a_n} \right|.$$

Since polynomials and the absolute value function are continuous, and any composition of continuous functions is continuous, the real-valued function $q : z \to |p(z)|$ on D is continuous, hence it assumes a minimum value. That is, there exists $c \in D$ such that $q(c) \le q(w)$ for all $w \in D$. (These properties of continuous functions weren't stated and proved until about fifty years after Argand's work; Remmert says that Argand would have thought them obvious, not needing proof.)

Argand showed $p(c) = 0$ by assuming the opposite and deriving a contradiction. Under that opposite assumption the function $h : z \to p(c)^{-1}\,p(z + c)$ would be a nonconstant polynomial. That is, there would exist integers k and n such that $1 \le k \le n$, and complex numbers $b_k, \ldots, b_n$ such that $b_k \ne 0$ and $h(z) = 1 + b_kz^k + \cdots + b_nz^n$ for all complex z. Find d such that $d^k = -1/b_k$. The polynomial $g : z \to b_{k+1}d^{k+1}z + \cdots + b_nd^nz^{n-k}$ is continuous and $g(0) = 0$, so there exists $\delta > 0$ such that $|g(t)| < ½$ for all t such that $0 \le t < \delta$. Select any t such that $0 < t < 1,\delta$ and let $w = dt + c$. Then

$$\begin{aligned}
|h(dt)| &= |1 + b_k(dt)^k + \cdots + b_n(dt)^n| \\
&= |1 + b_kd^kt^k + b_{k+1}d^{k+1}t^{k+1} + \cdots + b_nd^nt^n| \\
&= |1 - t^k + t^kg(t)| \le |1 - t^k| + |t^kg(t)| \\
&= 1 - t^k + t^k|g(t)| < 1 - t^k + ½\,t^k < 1 - ½\,t^k < 1
\end{aligned}$$

$$q(w) = |p(dt + c)| = |p(c)h(dt)| = |p(c)||h(dt)| < |p(c)| = q(c).$$

This would contradict the previous paragraph, which established $q(c) \le q(w)$.

Reference

Remmert, Reinhold. 1993. Vom Fundamentalsatz der Algebra zum Satz von Gelfand-Mazur. *Mathematische Semesterberichte* 40: 63–71.