Definitions and Nondefinability in Geometry: Pieri and the Tarski School[1]

James T. Smith
San Francisco State University

Introduction

In 1886 Mario Pieri became professor of projective and descriptive geometry at the Royal Military Academy in Turin. In 1888 he was also appointed assistant at the University of Turin. By the early 1890s he and Giuseppe Peano, colleagues at both institutions, were researching related questions about the foundations of geometry. During the next two decades, Pieri used, refined, and publicized Peano's logical methods in several major studies in this area. The present paper focuses on the history of two of them, Pieri's 1900 *Point and Motion* and 1908 *Point and Sphere* memoirs, and on their influence as the root of later work of Alfred Tarski and his followers. It emphasizes Pieri's achievements in expressing Euclidean geometry with a minimal family of undefined notions, and in requiring set-theoretic constructs only in his treatment of continuity. It is adapted from and expands on material in the 2007 study of Pieri by Elena Anne Marchisotto and the present author.

Although Pieri 1908 had received little explicit attention, during the 1920s Tarski noticed its minimal set of undefined notions, its extreme logical precision, and its use of only a restricted variety of logical methods. Those features permitted Tarski to adapt and reformulate Pieri's system in the context of first-order logic, which was only then emerging as a coherent framework for logical studies. Tarski's theory was simpler, and encouraged deeper investigations into the metamathematics of geometry.

In particular, Tarski and Adolf Lindenbaum pursued the study of definability, extending earlier work by the Peano School. They settled some questions about systems related to Pieri's, and during the 1930s showed that in the first-order context with variables ranging over points, Pieri's selection of ternary equidistance as the sole undefined relation for Euclidean geometry was optimal. No family of binary relations, however large, can serve as the sole undefined relations.

Tarski's work itself went mostly unpublished for decades, but began to attract further research during the 1950s. Tarski's followers have extended his methods to apply to other geometric theories as well as the Euclidean. The present paper concludes with a description of the 1990–1991 discovery by Victor Pambuccian: Euclidean geometry can be based on a single binary point relation if the underlying logic is strengthened.

[1] For the congress, *Giuseppe Peano and his School between Mathematics, Logic, and Interlingua*, 2–7 October 2008, in Turin. The author gratefully acknowledges inspiration by Elena Anne Marchisotto and suggestions from Victor Pambuccian.

Pasch

Moritz Pasch began his career around 1870 as an algebraic geometer, but changed his emphasis to foundations of analysis and geometry. To correct logical gaps in classical Euclidean geometry and in G. K. C. von Staudt's 1847 presentation of projective geometry, Pasch published in 1882 the first completely rigorous synthetic presentation of a geometric theory.

Pasch noted that, in contrast to earlier practice, he would discuss certain notions without definition. Determining which ones he actually left undefined requires close reading.[2] They are

- *point*,
- segment *between* two points,
- *coplanarity* of a point set,
- *congruence* of point sets.

Pasch defined all other geometric notions from those. For example, he called three points *collinear* if they are not distinct or one lies between the other two, and defined the *line* determined by two distinct points to be the set of points collinear with them. Pasch developed incidence and congruence geometry, extended it to projective space, then showed (section 20) how to select a polar system to develop Euclidean or non-Euclidean geometry. In spite of the abstract nature of his presentation, Pasch clearly indicated that he regarded his postulates as descriptions of the real world:

> In contrast to the propositions justified by proofs … there remains a group of propositions from which all others follow … based directly on observations … .[3]

Peano and Motions

Giuseppe Peano began intense study of fundamental principles of mathematics during the 1880s. His 1889 booklet on foundations of geometry contained some technical improvements over Pasch 1882. But more importantly, Peano departed from Pasch's then prevalent approach by divorcing that discipline from the study of the real world:

> Depending on the significance attributed to the undefined symbols … the axioms can be satisfied or not. If a certain group of axioms is verified, then all the propositions that are deduced from them will be equally true … .[4]

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[2] See the comments after the list of twenty-three axioms and theorems in Pasch 1882 section 1. Already in the introduction (*Einleitung*), Pasch discussed points without definition. Sections 1 and 2 begin by introducing betweenness and coplanarity. Pasch distinguished the notions of coplanar set (*ebene Fläche*) and plane (*Ebene*). Not until section 13 did he introduce congruence.

[3] Pasch 1882, 17: "Nach Ausscheidung der auf Beweise gestützten Sätze … bleibt eine Gruppe von Sätzen zurück, aus denen alle übrigen sich folgern lassen … ; diese sind unmittelbar auf Beobachtungen gegründet … ." See also Jané 2006, §6.

[4] Peano 1889, 24: "Dipendentemente dal significato attribuito ai segni non definiti … potranno essere soddisfatti, oppure no, gli assiomi. Se un certo gruppo di assiomi è verificato, saranno pure vere tutte le proposizioni che si deducono … ."

This freedom to consider various interpretations of the undefined notions, and the distinction between syntactic properties of symbols and their semantic relationships to the objects they denote, was essential for all later studies of definability.

In 1894 Peano introduced the use of *direct motion* to replace congruence as an undefined notion in Euclidean geometry. A geometric transformation, this sort of motion does not involve time. Figures can be defined as *congruent* if some direct motion maps one to the other.[5]

Pieri and Motions

After earning the doctorate in 1884 at Pisa, Mario Pieri began his research career in algebraic and differential geometry. Soon after Pieri's appointments in Turin, his colleague Corrado Segre suggested that Pieri translate Staudt 1847, the fundamental work on the projective geometry that underlies those areas.[6] Evidently Pieri, like Pasch, became intrigued with its logic: he returned to study it again and again. Pieri's senior colleague Giuseppe Peano was already investigating deep questions in foundations, and in 1890 himself repaired a lapse in Staudt's work. During the 1890s, Pieri became a preeminent member of the Peano School. A series of his papers culminated in Pieri 1898, the first complete axiomatization of projective geometry.

To frame this work and later axiomatic studies, Pieri used a *hypothetical-deductive system*. He and Alessandro Padoa introduced this technique explicitly to formalize Peano's idea that the undefined notions may be interpreted arbitrarily, as long as their interpretations satisfy the postulates. This approach was adopted in the many papers by postulate theorists during 1900–1925, in particular Huntington 1904, which precisely echoed the Italians' formulation. This has become today's version of the axiomatic method.[7]

Following Peano's lead, Pieri pursued deeply the use of direct motion as an undefined notion. His 1900 *Point and Motion* memoir was an axiomatization of a large part of elementary geometry, common to Euclidean and hyperbolic geometry, but independent of continuity considerations. He employed only two undefined notions, *point* and *direct motion*! The following definitions[8] were central:

5 With additional work, *indirect* motions can be defined, which relate *anticongruent* figures.

6 Pieri 1889 is an annotated translation of Staudt 1847.

7 Pieri 1900, *prefazione*; Pieri [1900] 1901, §III; Padoa [1900] 1901, introduction; Huntington 1904, 288–290. See also Jané 2006, §6, and Scanlan 2003, §2.

8 See Pieri 1900, P9§1, for the definition of collinearity; P28§1 and P7§3 for equidistance; P7§2 for midpoint; and P1,2,6§4 for betweenness.

- Three points are called *collinear* if they are fixed by some nontrivial motion.
- Points M,P are called *equidistant* from point O if some direct motion maps M to P but fixes O.
- A point is said to *lie midway between* two others if it is collinear with and equidistant from them.
- A point Q is said to *lie somewhere between* two points P,R if it lies midway between two points M,N such that M,P and N,P are equidistant from a point O midway between P,R. (Figure 1 displays this ingenious definition.)

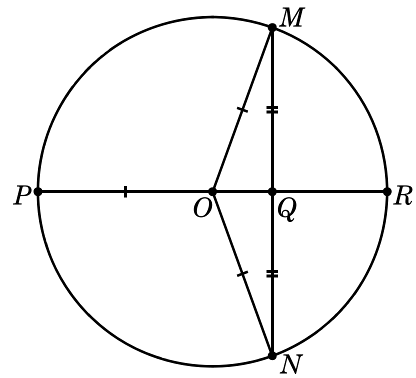


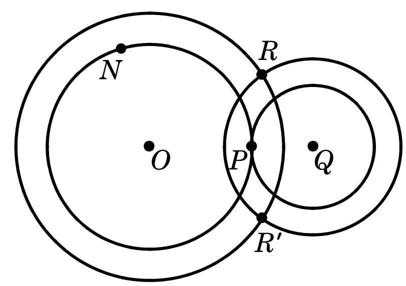
Figure 1

Pieri's 1900 definition of
Q *lying somewhere between* P,R

Pieri's *Point and Sphere* Memoir

After a long struggle, Mario Pieri finally obtained appointment as university professor in 1900, at Catania.[9] There he completed his 1908 *Point and Sphere* memoir, a full axiomatization of Euclidean geometry based solely on the undefined notions *point* and *equidistance* of two points N,P from a third point O, written $ON = OP$. Already in 1900 (preface, 176) he had suggested that this was possible. Pieri used the following definitions[10] (letters N–R may refer to points in figure 2):

- N is said to lie *on the sphere* P_O *through* P *about* O if $ON = OP$.



P is collinear with O, Q; R is not.

Figure 2

Pieri's 1908 definition of
collinearity

[9] See Marchisotto and Smith 2007, chapter 1, for details of Pieri's life and career.

[10] See Pieri 1908, P4§1, for the definition of sphere; P11,21§1 for collinearity; P45§1 for reflection about a point and for congruence of spheres; P31§4 for congruence of point pairs; P1,36§4 for isometry; and P27§7 for direct motion (there it is called "congruence"). Gruszczińki and Pietruszczak 2007 is a clear and concise exposition of most of Pieri's 1908 definitions.

- If $O \neq Q$, then P is called *collinear with* O,Q if P_O intersects P_Q only at P. (Pieri adapted this definition from Leibniz.[11])
- Q is called a *reflection of* O *over* P if P is collinear with O,Q and $PO = PQ$.
- Two spheres are called *congruent* if the points on one are related to those on the other by reflection over some single point.
- Point pairs O,P and Q,R are called *congruent* if R lies on a sphere about Q congruent to P_O.
- An *isometry* is a point transformation that preserves congruence of pairs.
- A *direct motion* is the composition of an isometry with itself.

Pieri then proceeded as in 1900. His axioms were frightfully complicated, but would now be called first-order, except for the continuity axioms. Moreover, he published all details of his proofs!

Tarski's System of Geometry

In 1926, starting on the path to become the world's top logician, Alfred Tarski was a university assistant and high-school teacher in Warsaw.[12] His research emphasized application of logical techniques to geometric problems. Studying axiomatic presentations of geometry, he adopted the refinements of the axiomatic method introduced by the Peano School. Tarski was also beginning to emphasize *first-order* logic, which avoids use of sets.[13] Tarski adapted the approach of Pieri's 1908 *Point and Sphere* memoir, which fit into that framework. Reporting a conversation with Tarski, Steven Givant wrote,

> Tarski was critical of Hilbert's axiom system ... [and] preferred Pieri's system [1908], where the logical structure and the complexity of the axioms were more transparent.

Tarski developed and presented his own axiomatization in a 1926–1927 Warsaw University course. In such contexts he also followed Pieri's practice of full disclosure of all proofs; but those are so cumbersome that fifty years passed before they became available to the public.[14]

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[11] Leibniz [1679] 1971, part IV, 185, 189.

[12] For biographical information about Tarski, consult Feferman and Feferman 2004.

[13] See Jané 2006, §10, and Scanlan 2003, §2. Tarski's [1927] 1983 treatment of the geometry of "solid" point sets was based on Pieri 1908. Moreover, Tarski cited Huntington 1913, which reprised the description of hypothetical-deductive systems in Huntington 1904, mentioned earlier. Presburger 1930, footnote 4, mentioned Tarski's use of first-order logic.

[14] Givant 1999, 50; Hilbert [1899] 1971. That Tarski's system is closest in spirit to Pieri's, or patterned after it, is maintained in Tarski and Givant 1999, §3, and Szczerba 1986, 908. The present author's personal testament to Tarski's regard for Pieri is recorded in Marchisotto and Smith 2007, 357. It seems unknown whether Tarski consulted the original Pieri 1908 memoir or its Polish translation Pieri 1915, or even why that translation was published. Many of the proofs of Tarski's results were polished and published for the first time in his student H. N. Gupta's huge 1965 dissertation.

Tarski's undefined notions were *point* and two relations, *congruence* of two point pairs and *betweenness* of a triple.[15] With these slightly more complex undefined notions, Tarski was able to greatly simplify Pieri's 1908 axioms. Tarski's axioms were two-dimensional but easily modifiable for use in three dimensions without loss of simplicity. As continuity axioms, he used all first-order instances of Pieri's second-order axiom. All Tarski's axioms except those for continuity had $\forall\exists$ form, with all universal quantifiers preceding all existential quantifiers at the beginning. Their total length was less than that of Pieri's single most complicated axiom.[16] Tarski proved that the models of his axioms are the structures isomorphic with coordinate planes over real-closed ordered fields.[17]

Tarski's system was not broadly publicized until his [1957] 1959 summary; his proofs remained unpublished commercially until Schwabhäuser, Szmielew, and Tarski 1983. But the formulation of the system enabled much deeper research into provability, decidability, and definability in geometry.

Nondefinability

In 1904, Oswald Veblen proposed an alternative to Mario Pieri's 1908 *Point and Sphere* axiomatization that regarded only *point* and *betweenness* as undefined. His axioms were much simpler than Pieri's. Veblen followed Pasch 1882 in using a projective polar system to define Euclidean congruence, and hence equidistance. In 1907, however, Federigo Enriques noted that Veblen's polar system was not uniquely determined: it seemed also to be an undefined notion.

Settling that dispute required a precise definition of *definition* in Euclidean geometry. This was achieved by first adopting as standard some first-order axiom system, such as Alfred Tarski's. If ν is a notion and Φ a family of notions defined in that system, then a first-order phrase involving only the notions in Φ should be called a *definition* of ν in terms of Φ if it provably characterizes ν in the standard system.[18] In 1935, after considering definitions in general, Tarski noted that *betweenness* cannot serve as the sole undefined relation in a first-order axiomatization of Euclidean geometry with variables ranging over points. He suggested an argument using a technique introduced in 1900 by Alessandro Padoa, another leading member of the Peano School: any affine

[15] Oswald Veblen had based Euclidean geometry on these three notions in 1911.

[16] The version of the Pasch axiom in Pieri 1908, P13§3, has form $\forall\exists\forall\exists$. In 2008, Victor Pambuccian showed how to construct an axiom system equivalent to Pieri's, using just the single undefined ternary equidistance relation, in which all axioms except those for continuity have $\forall\exists$ form.

[17] An ordered field F is called *Euclidean* if its nonnegative elements are all squares, and *real-closed* if it is Euclidean and every polynomial over F with odd degree has a root in F.

[18] Tarski sometimes admitted use of notions beyond first-order, but that does not affect the discussion in the present paper. In 1941 Rudolf Carnap summarized a verbal comment by Tarski: "The Warsaw logicians saw a system like PM (but with simple type theory) as the obvious system form. This restriction influenced strongly all the disciples; including Tarski until [about 1935]." (Mancosu 2005, 335)

transformation that is not a similarity would preserve betweenness, and thus also any notion defined by a first-order phrase solely in terms of betweenness; but it would not preserve equidistance or congruence.[19]

That same year, Adolf Lindenbaum and Tarski remarked[20] that Pieri's selection of ternary equidistance as the sole undefined relation was optimal: no family Φ, however large, of binary relations among points can serve as the family of all undefined relations. Those would have to be definable in the standard system, hence invariant under all similarities. But it is easy to show that if ρ is a binary relation among points, different from the empty, equality, inequality, and universal relations, then

$$(\exists P,Q)[\rho PP \ \& \ \neg\rho QQ] \lor (\exists P,Q,R,S)[P \neq Q \ \& \ R \neq S \ \& \ \rho PQ \ \& \ \neg\rho RS].$$

Since any pair P,Q of distinct points can be mapped to any other by some similarity, only those four exceptional binary relations are invariant under all similarities: Φ could contain only those four. They are in fact invariant under *all* transformations, as are any relations defined solely from them. *Some* transformations, however, fail to preserve betweenness and equidistance. Thus neither of those can be defined solely in terms of relations in Φ.

Tarski's work has led to related studies, many reported in Schwabhäuser, Szmielew, and Tarski 1983: for example, what other ternary relations suffice as the sole undefined relation? More recently, Victor Pambuccian has investigated the effect of strengthening the underlying logic to permit conjunctions $\&_{m,n}\varphi_{m,n}$ of infinite families of open sentences $\varphi_{m,n}$ that depend on natural numbers m,n.[21] He discovered a startling fact in 1990–1991: with that logic, a single binary relation v can be used as the sole undefined relation! This relation vPQ holds for points P,Q just when the distance $PQ = 1$. Pambuccian considered for each m,n the auxiliary relation $v_{m,n}PQ$ that says $PQ = m/2^n$, which can be defined solely in terms of vPQ by a complicated first-order phrase that describes some familiar geometric constructions. He then proved that P,Q is congruent to another point pair R,S just when

$$\begin{aligned}\&_{m,n}\big[&[\exists T[v_{m,n}PT \ \& \ v_{m,n}QT] \Rightarrow \exists U[v_{m,n}RU \ \& \ v_{m,n}SU]] \\ &\& \ [\exists T[v_{m,n}RT \ \& \ v_{m,n}ST] \Rightarrow \exists U[v_{m,n}PU \ \& \ v_{m,n}QU]]\big].\end{aligned}$$

The first implication fails for some m,n just when $PQ < RS$. (See figure 3.) The displayed open sentence is a definition, in the strengthened logical system, of congruence of point pairs. That yields a definition of Pieri's single undefined relation, ternary equidistance, solely in terms of v. Therefore, with the new logic, v can also serve as the single undefined relation. But the geometry is not new: the constructions that

[19] Tarski [1935] 1983, 299–307; Padoa [1900] 1901, 322, or Van Heijenoort 1967, 122.

[20] Lindenbaum and Tarski [1935] 1983, 388–389. They did not mention Pieri explicitly.

[21] For information on logic with infinite conjunctions, consult Karp 1964.

Pambuccian employed in 1990–1991 to define $v_{m,n}$ in terms of v had already been used by Pieri in 1908 to analyze the continuity of a line![22]

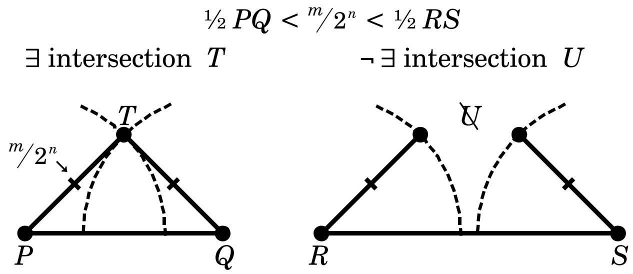

Figure 3
Pambuccian's 1990–1991
definition of $PQ < RS$

Conclusion

The pioneering work of Giuseppe Peano during 1889–1894 on the logic underlying geometry, and on the use of direct motion as a fundamental geometric idea, led to Pieri's detailed 1900 and 1908 axiomatizations of geometry. Pieri formalized his presentations as hypothetical-deductive systems, used minimal sets of undefined notions, relied on set theory only for continuity considerations, and published all the details of his proofs. During the 1920s Alfred Tarski adapted Pieri's approach to achieve a surprisingly efficient first-order axiomatization of Euclidean geometry, which has become standard. It allowed Tarski to formulate in the 1930s a theory of first-order definitions, with which he proved that Pieri's axiomatization was optimal. In the 1990s, Victor Pambuccian, using geometry that would have been familiar to Pieri, showed that some greater economy could be achieved, but only by strengthening the underlying logic. Research in this direction continues today.

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[22] Pieri 1908, §1V, P18, and §VIII, P21ff. In 2001, Apoloniusz Tyszka improved Pambuccian's 1990–1991 result. Tyszka showed that point pairs P, Q and R, S are congruent just when a certain existential closure of a countably infinite conjunction holds, whose components are countably infinite disjunctions of finite conjunctions of formulas involving just variables, the relation symbol v, and the equality symbol. Such *positive existential* conditions have highly desirable logical properties.

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