BINARY ARITHMETIC

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A *numeral* is the name of a number. We construct *base* n numerals for the natural numbers by counting as usual with the first n digits starting at 0. For *binary* numerals, $n = 2$:

Decimal numerals	*Binary numerals*
0	0
1	1
2	10
3	11
4	100
5	101
6	110
7	111
8	1000
9	1001
10	1010
11	1011
12	1100
13	1101
⋮	⋮

The two binary digits 0 and 1 are called *bits*.[1] Often we use a subscript to indicate the base:

$$1504_{10} = 1\times10^3 + 5\times10^2 + 0\times10^1 + 4\times10^0$$
$$1101_2 = 1\times2^3 + 1\times2^2 + 0\times2^1 + 1\times2^0.$$

(We write the base and the exponents with decimal numerals.)

You can use equations like the previous one to convert from binary to decimal numerals:

$$1101_2 = 1\times2^3 + 1\times2^2 + 0\times2^1 + 1\times10^0 = 8 + 4 + 0 + 1 = 13.$$

[1] This term is due to John Tukey. Click here for his biographical sketch and portraits on the St. Andrews website.

Decimal to binary conversion requires repeated short division with remainders:

2)13		
2) 6	remainder	1
2) 3	remainder	0
2) 1	remainder	1
0	remainder	1

Read the list of remainders *upward*.

Adding binary numerals is like adding decimals:

binary	decimal
101011	43
+1011	+11
= 110110	= 54

Note the rhythm of the carries: 1 + 1 = 0 carry 1, 1 + 1 + 1 = 1 carry 1. Adding columns of more than two binary numerals is usually impractical because of the profusion of carries.

You can *subtract* binary numerals as you subtract decimals, but borrowing is perplexing:

binary	decimal
01001	
1~~10110~~	54
–1011	–11
= 101011	= 43

To *multiply* a binary numeral by 2^p you just shift it left p places, filling on the right with zeros. Multiplication in general consists of shifting and adding:

```
     101011
     ×1011
    -------
     101011
    101011
  101011
  ---------
  111011001
```

You can also think of multiplying a binary numeral by 2^p as moving the binary point *rightward* p places. *Dividing* by 2^p moves the point *leftward* p places. Long division works as it does with decimal numerals:

```
      .1
  10)1.0
```

$(\frac{1}{2})_{10} = (.1)_2$

```
      .0101
  11)1.0000
      11
     ----
       100
        11
       ---
          ⋱
```

$(\frac{1}{3})_{10} = (.\overline{01})_2$

$(\frac{1}{4})_{10} = (.01)_2$

```
       .00110011
  101)1.00000000
       101
      ----
        110
        101
        ---
          1000
           101
          ----
             110
             101
             ---
                ⋱
```

$(\frac{1}{5})_{10} = (.\overline{0011})_2$

For some fractions, the decimal expansion terminates but the binary expansion does not.

Exercises

1. Add these binary numerals by hand:

```
  1011 0101 0001 1101
+ 0110 1001 1110 1001
```

2. Convert the two addends in exercise 1 and their sum to decimal and use the results to check your addition in exercise 1.

3. Subtract these binary numerals by hand and verify by hand addition:

```
  1011 0101 0001 1101
– 0110 1001 1110 1001
```

4. Multiply these binary numerals by hand:

```
  1011
×1101
```

5. Convert the two factors in exercise 4 and their product to decimal and check your multiplication in exercise 4.

6. Divide these binary numerals by hand, obtaining quotient and remainder: 1011 1101 1100 0011 / 1100 1001.

7. Convert the dividend, divisor, quotient, and remainder in exercise 6 to decimal and check your division in exercise 6.

8. Convert the decimal numeral 123456 to binary by hand.