BASIC SET THEORY

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These notes outline some set theory on which many parts of mathematics are based.

Sets

The notions *object*, *set*, and *membership* are used in this theory without definition. The expression $x \in X$ indicates that the object x is a member of the set X. Any object with a member is a set, and sets are considered objects. Sometimes it's assumed that sets are the only objects, but not in this outline.

Notation

Using gaudier letters for sets than for their members, as in $g \in G \in \mathcal{G}$, often enhances clarity (but sometimes isn't practical). These abbreviations are also useful:

$\in$	. . . is a member of	&	. . . and
$\notin$	. . . is not a member of	$\vee$	. . . or
$=$	. . . . equals	$\neg$	. . . not
$\neq$	. . . . does not equal	$\Rightarrow$	. . . if … then …
$\forall$	. . . for all	$\Leftrightarrow$	. . . if and only if
$\exists$	. . . for some		

Equality

Two sets are *equal* if and only if they have the same members:

$$X = Y \Leftrightarrow \forall t\,[\,t \in X \Leftrightarrow t \in Y\,].$$

That is the *extensionality principle*.

Extension

Frequently, a set X is described by a statement of the form

$$t \in X \Leftrightarrow \Phi \qquad (*)$$

where Φ is a condition involving t. For example, in calculus you often consider intervals of real numbers:

$$t \in [a,b] \Leftrightarrow a \leq t \ \& \ t \leq b.$$

If (*) holds, then X is called the *extension* of Φ. It's appropriate to call X *the* extension, because, by the extensionality principle, you can deduce $X = Y$ from (*) and the similar statement $t \in Y \Leftrightarrow \Phi$. Since X is uniquely determined when (*) holds, the notation

$$X = \{t : \Phi\}$$

is common. It's read, "X is the set of all t such that Φ." For the previous example,

$$[a,b] = \{ t : a \leq t \ \& \ t \leq b \}.$$

Each set X is the extension of *some* condition Φ —for example, the condition $t \in X$. That is,

$$t \in X \Leftrightarrow t \in X \qquad X = \{t : t \in X\}.$$

However, there are conditions Φ that have no extension—that is, for which there's no set X such that (*) holds. In 1902, Bertrand Russell discovered the most celebrated such condition: $t \notin t$. If that had an extension X, then $X \in X \Leftrightarrow X \notin X$, contradiction!

One of the most important problems in foundations of mathematics is to determine which conditions have extensions. This outline, however, doesn't attack that question. Frequently, it introduces new sets X as extensions of certain conditions. The assumptions that these particular conditions have extensions have never led to contradiction.

Separation

One type of condition always has an extension: a condition applying only to members of a previously given set. That is, to each set Y and each condition Φ corresponds a set X whose elements are those members t of Y that satisfy Φ:

$$X = \{t : t \in Y \ \& \ \Phi\}, \text{ abbreviated } \{t \in Y : \Phi\}.$$

This is the *separation principle*. It implies, for example, that the condition $t = t$ has no extension: if there existed a set V such that $t \in V \Leftrightarrow t = t$, then Russell's condition would have an extension, namely $\{t \in V : t \notin t\}$. This result can also be phrased, *there's no "universal" set that contains all objects*.

Inclusion

A set X is said to be *included in* a set Y —or called *a subset of* Y —if each member of X belongs also to Y:

$$X \subseteq Y \Leftrightarrow \forall t[t \in X \Rightarrow t \in Y].$$

This concept has the following properties:

$X \subseteq X$ —*reflexivity*
$X \subseteq Y \;\&\; Y \subseteq X \Rightarrow X = Y$ —*weak antisymmetry*
$X \subseteq Y \;\&\; Y \subseteq Z \Rightarrow X \subseteq Z$ —*transitivity*.

Power Set

To each set X corresponds a set $\mathcal{P}X$, called the *power set* of X, whose members are the subsets of X:

$$\mathcal{P}X = \{S : S \subseteq X\}.$$

Empty Set

The condition $t \neq t$ has an extension, called the *empty set* ϕ:

$$\phi = \{t : t \neq t\} \qquad \forall t[t \notin \phi].$$

By the extensionality principle, ϕ is the only set with no members. It's a subset of every set.

Singletons

To each object x corresponds a set $\{x\}$, called *singleton* x, whose sole member is x:

$$\{x\} = \{t : t = x\}.$$

Pairs

To any objects x and y corresponds a set $\{x,y\}$, called a *pair*, whose only members are x and y:

$$\{x,y\} = \{t : t = x \vee t = y\}.$$

Notice that

$$\{x, y\} = \{y, x\} \qquad \{x, x\} = \{x\}.$$

Triples, quadruples, etc., could be introduced the same way, but more comprehensive methods will be presented later.

Ordered Pairs

To any two objects x and y corresponds an object $<x, y>$, called an *ordered pair*. For any objects x, y, x', and y',

$$<x, y> = <x', y'> \Leftrightarrow x = x' \ \& \ y = y'.$$

Ordered triples and quadruples, etc., could be introduced the same way, but it's easier to define

$$<x, y, z> = <<x, y>, z>$$

and extend that idea to quadruples, etc.

Cartesian Product

To any sets X and Y corresponds a set $X \times Y$, called their *Cartesian product*, whose members are the ordered pairs whose first and second entries belong to X and Y:

$$<x, y> \in X \times Y \Leftrightarrow x \in X \ \& \ y \in Y.$$

Cartesian products of three or more sets are introduced as follows:

$$X \times Y \times Z = (X \times Y) \times Z$$
$$<x, y, z> \in X \times Y \times Z \Leftrightarrow x \in X \ \& \ y \in Y \ \& \ z \in Z.$$

Relations

A *relation* between two sets X and Y is a subset of $X \times Y$. Thus ϕ and $X \times Y$ itself are relations between X and Y. This abbreviation is commonly used for relations R:

$$x \, R \, y \Leftrightarrow <x, y> \in R.$$

A relation between X and itself is called a relation *on* X.

Domain and Range

To each relation R correspond two sets, called its *domain* and *range*, whose members are the first and second entries of the members of R:

$$x \in \operatorname{Dom} R \;\Leftrightarrow\; \exists y\,[x\,R\,y] \qquad\qquad y \in \operatorname{Rng} R \;\Leftrightarrow\; \exists x\,[x\,R\,y].$$

Converse

To each relation R between sets X and Y corresponds a relation $\check{R}$ between Y and X, called the *converse* of R, such that

$$y\,\check{R}\,x \;\Leftrightarrow\; x\,R\,y.$$

Relative Product

If R is a relation between sets X and Y and S is a relation between sets Y and Z, then their *relative product* is the relation $R|S$ defined as follows:

$$x\,(R|S)\,z \;\Leftrightarrow\; \exists y\,[x\,R\,y \;\;\&\;\; y\,S\,z].$$

For example, if R is the relation of person to parent and S that of sibling to brother, then $R|S$ is the relation of person to uncle.

The following *associative law* is fundamental: for any R and S as described and any relation Q between sets W and X,

$$(Q|R)|S = Q|(R|S).$$

Proof. Suppose $w\,((Q|R)|S)\,z$. Then

$$\exists y\,[w\,(Q|R)\,y \;\;\&\;\; y\,S\,z\,]$$

↙

$$\exists x\,[w\,Q\,x \;\;\&\;\; x\,R\,y]$$

↘ ↙

$$x\,(R|S)\,z$$

↘ ↙

$$w\,(Q|(R|S))\,z.$$

Thus the left hand side of the associative law equation is included in the right hand side. You can demonstrate the reverse inclusion similarly.

The associative law permits the abbreviation $Q|R|S$ for $(Q|R)|S$ or $Q|(R|S)$. The following law is also important: for any relations R and S as described earlier,

$$T = R|S \;\Rightarrow\; \check{T} = \check{S}|\check{R}.$$

You can supply the proof.

Identity

To each set X corresponds the *identity* relation I_X on X:

$$x\, I_X\, x' \;\Leftrightarrow\; x = x' \;\;\&\;\; x \in X.$$

For each relation R between X and a set Y,

$$I_X|R = R = R|I_Y.$$

Image

To each relation R and each set A corresponds a subset $R[A]$ of $\operatorname{Rng} R$ called the *image of* A *under* R:

$$y \in R[A] \;\Leftrightarrow\; \exists x[x \in A \;\;\&\;\; x\, R\, y].$$

For each relation S,

$$(R|S)[A] = S[R[A]].$$

Functions

A relation F is called a *function from* a set X *to* a set Y if $\operatorname{Dom} F = X$, $\operatorname{Rng} F \subseteq Y$ and for all x, y,

$$x\, F\, y \;\;\&\;\; x\, F\, y' \Rightarrow y = y'.$$

That is written $F : X \to Y$. If $F : X \to Y$ and $x \in X$ then there is a unique $y \in Y$ such that $x\, F\, y$. This is written $F : x \mapsto y$. That y is called the *value of* F *at* x, and it is denoted by $F(x)$ or F_x. By the extensionality principle, two functions F and G are equal if and only if they have the same domain X and $F(x) = G(x)$ for all $x \in X$.

The functions from a set X to a set Y constitute a set Y^X. Here are some rules for manipulating these function sets:

$$Y^{\phi} = \{\phi\} \qquad X \neq \phi \;\Rightarrow\; \phi^X = \phi \qquad X \subseteq Y \;\Rightarrow\; I_X \in Y^X.$$

A function $f : X \to X$ is often called a *singulary operation* on X; a function $g : X \times X \to X$ is often called a *binary operation* on X. A function is called *constant* if its range is a singleton.

Composition

The *composition* of functions $F : X \to Y$ and $G : Y \to Z$ is the function $G \circ F = F|G$ from X to Z. The *associativity law* holds: if also $E : W \to X$, then

$$(G \circ F) \circ E = G \circ (F \circ E).$$

That permits the abbreviation $G \circ F \circ E$ for either of these compositions. The following manipulation rules hold:

$$x \in X \;\Rightarrow\; (G \circ F)(x) = G(F(x))$$
$$A \subseteq X \;\Rightarrow\; (G \circ F)[A] = G[F[A]]$$
$$I_Y \circ F = F = F \circ I_X.$$

Injections

If $F : X \to Y$ and $\breve{F}$ is a function, then we say that $F : X \to Y$ *injectively*, and call F an *injection*. The following rules are helpful:

$\phi : \phi \to Y$ injectively

$X \subseteq Y \;\Rightarrow\; I_X : X \to Y$ injectively

$F : X \to Y$ injectively & $G : Y \to Z$ injectively
$\quad\Rightarrow\; G \circ F : X \to Z$ injectively.

Surjections

If $F : X \to Y$ and $\operatorname{Rng} F = Y$, then we say that $F : X \to Y$ *surjectively*, and call F a *surjection*. The following rules hold:

$\phi : \phi \to \phi$ surjectively

$F : X \to Y$ surjectively & $G : Y \to Z$ surjectively
$\quad\Rightarrow\; G \circ F : X \to Z$ surjectively.

Bijections

If $F : X \to Y$ injectively and surjectively, then we say that $F : X \to Y$ *bijectively* and call F a *bijection*. Here are useful rules:

$I_X : X \to X$ bijectively

$F : X \to Y$ bijectively $\Rightarrow$ $\breve{F} : Y \to X$ bijectively

$F : X \to Y$ bijectively & $G : Y \to Z$ bijectively
$\quad\Rightarrow\; G \circ F : X \to Z$ bijectively.

A bijection from X to itself is called a *permutation* of X. The set $X!$ of all permutations of X is called the *symmetric group* on X.

Inverse

If $F: X \to Y$ bijectively, then $\breve{F}$ is called the *inverse* of F and denoted by F^{-1}. Here's its most important property:

$$F^{-1} \circ F = I_X \qquad F \circ F^{-1} = I_Y.$$

Proof. To show $I_X \subseteq F^{-1} \circ F$, let $x \in X$ and define $y = F(x)$. Then $x\,F\,y$, so that $y\,F^{-1}\,x$ and hence $x\,F|F^{-1}\,x$, i.e. $<x,x> \in F^{-1} \circ F$. To show $F^{-1} \circ F \subseteq I_X$, let $<x,x'> \in F^{-1} \circ F$, i.e. $x\,F|F^{-1}\,x'$. Then there exists y such that $x\,F\,y$ and $y\,F^{-1}\,x'$. But $x\,F\,y$ implies $y\,F^{-1}\,x$, and thus $x = x'$ because F^{-1} is a function. The proof that $F \circ F^{-1} = I_Y$ is similar.

Here's an important property of the inverse, complementary to the previous one:

$$G : Y \to X \;\&\; G \circ F = I_X \Rightarrow G = F^{-1}$$
$$G : Y \to X \;\&\; F \circ G = I_Y \Rightarrow G = F^{-1}.$$

Proof. Suppose $G \circ F = I_X$; then $G = G \circ I_Y = G \circ (F \circ F^{-1}) = (G \circ F) \circ F^{-1} = I_X \circ F^{-1} = F^{-1}$. The second result is proved similarly.

Finally, if $F : X \to Y$ and $G : Y \to Z$ bijectively, then

$$(G \circ F)^{-1} = F^{-1} \circ G^{-1}.$$

Union

To any sets X and Y corresponds a set $X \cup Y$ called their *union*, whose members are the members of X and those of Y:

$$t \in X \cup Y \Leftrightarrow t \in X \vee t \in Y.$$

The union is their "least upper bound" in the following sense:

$$X, Y \subseteq X \cup Y \qquad X \subseteq Z \;\&\; Y \subseteq Z \Rightarrow X \cup Y \subseteq Z.$$

These *commutative* and *associative laws* hold:

$$X \cup Y = Y \cup X \qquad X \cup (Y \cup Z) = (X \cup Y) \cup Z.$$

Associativity permits the abbreviation $X \cup Y \cup Z$ for either side of this equation.

Intersection

To any sets X and Y corresponds a set $X \cap Y$ called their *intersection*, whose members are the elements common to X and Y:

$$t \in X \cap Y \Leftrightarrow t \in X \ \& \ t \in Y.$$

The intersection is their "greatest lower bound" in the following sense:

$$X \cap Y \subseteq X, Y \qquad\qquad Z \subseteq X \ \& \ Z \subseteq Y \Rightarrow Z \subseteq X \cap Y.$$

If $X \cap Y = \phi$, then X and Y are called *disjoint*. These *commutative*, *associative*, and *distributive laws* hold:

$$X \cap Y = Y \cap X \qquad\qquad X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$$
$$X \cap (Y \cap Z) = (X \cap Y) \cap Z \qquad\qquad X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z).$$

Associativity permits the abbreviation $X \cap Y \cap Z$ for either side of the second equation.

Union, Continued

To any set $\mathscr{X}$ corresponds a set $\bigcup\mathscr{X}$ called its *union*, whose members are the members of the members of $\mathscr{X}$:

$$t \in \bigcup\mathscr{X} \Leftrightarrow \exists X[X \in \mathscr{X} \ \& \ t \in X].$$

The union of $\mathscr{X}$ is its "least upper bound" in the following sense:

$$\forall X[X \in \mathscr{X} \Rightarrow X \subseteq \bigcup\mathscr{X}] \qquad\qquad \forall X[X \in \mathscr{X} \Rightarrow X \subseteq Y] \Rightarrow \bigcup\mathscr{X} \subseteq Y.$$

For any sets X and Y, $\bigcup\{X, Y\} = X \cup Y$.

Intersection, Continued

To any nonempty set $\mathscr{X}$ corresponds a set $\bigcap\mathscr{X}$ called its *intersection*, whose members are the elements common to all members of $\mathscr{X}$:

$$t \in \bigcap\mathscr{X} \Leftrightarrow \forall X[X \in \mathscr{X} \Rightarrow t \in X].$$

The intersection of $\mathscr{X}$ is its "greatest lower bound" in the following sense:

$$\forall X[X \in \mathscr{X} \Rightarrow \bigcap\mathscr{X} \subseteq X] \qquad\qquad \forall X[X \in \mathscr{X} \Rightarrow Y \subseteq X] \Rightarrow Y \subseteq \bigcap\mathscr{X}.$$

For any sets X and Y, $\bigcap\{X, Y\} = X \cap Y$.

Union and Intersection, Continued

Often you'll be interested in the union or intersection of the range of a function X with domain I. This notation is common:

$$\bigcup_{i\in I} X_i = \{t : (\exists i \in I)[t \in X_i]\} \qquad \bigcap_{i\in I} X_i = \{t : (\forall i \in I)[t \in X_i]\}.$$

When the set I is clear from the context, these are usually abbreviated as $\bigcup_i X_i$ and $\bigcap_i X_i$. This notation simplifies the statements of many rules—for example, the *distributive laws*

$$A \cap \bigcup_i X_i = \bigcup_i (A \cap X_i) \qquad A \cup \bigcap_i X_i = \bigcap_i (A \cup X_i).$$

Relative Complement

In this paragraph, all sets are assumed to be subsets of a single set U. To each such set X corresponds a set $-X$, its *complement* (relative to U), whose members are those elements of U not in X:

$$-X = \{t \in U : t \notin X\}.$$

These rules hold:

$-U = \phi$	$-\phi = U$	
$--X = X$		—*double negation*
$X \cup -X = U$	$X \cap -X = \phi$	
$X \subseteq Y \Leftrightarrow -Y \subseteq -X$		—*contraposition*
$-\bigcup_{i\in I} X_i = \bigcap_{i\in I} (-X_i)$	$-\bigcap_{i\in I} X_i = \bigcup_{i\in I} (-X_i)$	—*de Morgan*

Natural Numbers

There's a set $\mathbb{N}$ whose members are called *natural numbers*. Among its members is ϕ, which in this context is called *zero* and written 0. There's a bijection

$$S : \mathbb{N} \to \{n \in \mathbb{N} : n \neq 0\}$$

called the *successor* operation, which satisfies the *first principle of recursive proof*:

$$0 \in X \subseteq \mathbb{N} \;\; \& \;\; \forall n [n \in X \Rightarrow S(n) \in X] \Rightarrow X = \mathbb{N}.$$

From these considerations follows— by a complicated argument—the *first principle of recursive definition*:

given any set Y, any $y \in Y$, and any function $G : Y \to Y$,
there's a unique function $F : \mathbb{N} \to Y$ such that
$F(0) = y \;\&\; \forall n[n \in \mathbb{N} \Rightarrow F(S(n)) = G(F(n))]$.

Binary *sum* and *product* operations $+$ and $\cdot$ and an order relation $\leq$ on $\mathbb{N}$ are defined, and their usual properties proved, following standard recursive methods. In particular, 1 is defined as $S(0)$, so that $S(n) = n + 1$ for all $n \in \mathbb{N}$; and 2 is defined as $1 + 1$.

A *second principle of recursive proof* is sometimes handier than the first:

every nonempty $X \subseteq \mathbb{N}$ contains a member w such that $w \leq x$ for all $x \in X$.

There's a corresponding *second principle of recursive definition*:

given any set Y and any function $G : \mathbb{N} \times \mathscr{P}(\mathbb{N} \times Y) \to Y$,
there's a unique $F : \mathbb{N} \to Y$ such that for each $n \in \mathbb{N}$,
$\forall n[n \in \mathbb{N} \Rightarrow F(n) = G(<n, \{<m, F(m)> : m \in \mathbb{N} \;\&\; m < n\}>)]$.

Integers

Following standard algebraic procedures, *integers* are defined as certain sets of ordered pairs of natural numbers, and the familiar arithmetic operations are constructed for them. They form an ordered integral domain $\mathbb{Z}$ in which each nonempty set of nonnegative elements has a minimum element. All such domains are isomorphic.

Rational Numbers

Again following standard algebraic procedures, *rational numbers* are defined as certain sets of ordered pairs of integers, and the familiar arithmetic operations are constructed for them. They form a prime ordered field $\mathbb{Q}$. All such fields are isomorphic.

Real Numbers

Following standard analytic procedures, *real numbers* are defined as certain sets of sequences of rational numbers—i.e. certain sets of functions from $\mathbb{N}$ to $\mathbb{Q}$ —and the familiar arithmetic operations are constructed for them. They form a complete ordered field $\mathbb{R}$. All such fields are isomorphic. (Alternative definitions of real numbers as certain sets of rational numbers or certain sequences of integers are common.)

Complex Numbers

Following a standard algebraic procedure, *complex numbers* are defined as pairs of real numbers, and the familiar arithmetic operations are constructed for them. They form an algebraically closed field $\mathbb{C}$.

Trivial questions

1. $\mathscr{P}\phi = ?$ $\mathscr{P}\{\phi\} = ?$
2. $\{x,x\} = ?$
3. $\phi \times X = ?$ $X \times \phi = ?$
4. $\mathrm{Dom}\,\phi = ?$ $\mathrm{Rng}\,\phi = ?$
5. $\mathrm{Dom}\,\{<x,y>\} = ?$ $\mathrm{Rng}\,\{<x,y>\} = ?$
6. $\mathrm{Dom}\,(X \times Y) = ?$ $\mathrm{Rng}\,(X \times Y) = ?$
7. $\breve{\phi} = ?$ $R = \{<x,y>\} \Rightarrow \breve{R} = ?$
8. $R = X \times Y \Rightarrow \breve{R} = ?$ $R = \breve{S} \Rightarrow \breve{R} = ?$
9. $\phi|R = ?$ $R|\phi = ?$
10. $\{<x,y>\}|\{<y,z>\} = ?$
11. $(X \times Y)|(Y \times Z) = ?$ *Careful*!
12. $\mathrm{Dom}(R|S) \subseteq \mathrm{Dom}(?)$ $\mathrm{Rng}(R|S) \subseteq \mathrm{Rng}(?)$
13. $I_\phi = ?$ $I_{\{x\}} = ?$
14. $\phi[A] = ?$ $R[\phi] = ?$
15. $R[\mathrm{Dom}\,R] = ?$ $\breve{R}[\mathrm{Rng}\,R] = ?$
16. $X \cup \phi = ?$ $X \cap \phi = ?$
17. $X \cup X = ?$ $X \cap X = ?$
18. $\{x\} \cup \{y\} = ?$ $\{x\} \cap \{y\} = ?$
19. $X \cup Y = Y \Leftrightarrow ?$ $X \cap Y = Y \Leftrightarrow ?$
20. $X \cup (X \cap Y) = ?$ $X \cap (X \cup Y) = ?$
21. $I_X[A] = ?$
22. $\bigcup\phi = ?$
23. $\bigcup\{X\} = ?$ $\bigcap\{X\} = ?$
24. $A \cup \bigcup_{i \in I} X_i = \bigcup_{i \in I}(?)$ $A \cap \bigcap_{i \in I} X_i = \bigcap_{i \in I}(?)$

Routine exercises

1. Prove $\forall S[x \in S \Rightarrow y \in S] \Rightarrow x = y$.

2. Prove $X \subseteq Y \Rightarrow \mathscr{P}X \subseteq \mathscr{P}Y$.

3. Prove $\{\{x\},\{x,y\}\} = \{\{x'\},\{x',y'\}\} \Rightarrow x = x'$ & $y = y'$ (Kuratowski, 1921). The notion of ordered pair can be defined this way.

4. Prove $X \times Y = Y \times X \Leftrightarrow \phi = X \vee X = Y \vee Y = \phi$.

5. Suppose $X \subseteq X'$. What can you say about the relationship of

   a. $X \times Y$ and $X' \times Y$, $Y \times X$ and $Y \times X'$?
   b. $R[X]$ and $R[X']$?
   c. $X \cup Y$ and $X' \cup Y$, $Y \cup X$ and $Y \cup X'$?
   d. $X \cap Y$ and $X' \cap Y$, $Y \cap X$ and $Y \cap X'$?
   e. $\bigcup X$ and $\bigcup X'$?
   f. $\bigcap X$ and $\bigcap X'$?

6. Suppose R and R' are relations and $R \subseteq R'$. What can you say about the relationship of

   a. $\operatorname{Dom} R$ and $\operatorname{Dom} R'$, $\operatorname{Rng} R$ and $\operatorname{Rng} R'$?
   b. $\breve{R}$ and $\breve{R}'$?
   c. $R|S$ and $R'|S$, $S|R$ and $S|R'$?
   d. $R[A]$ and $R'[A]$?

7. Suppose $\forall i[X_i \subseteq Y_i]$. What can you say about the relationship of

   a. $\bigcup_i X_i$ and $\bigcup_i Y_i$?
   b. $\bigcap_i X_i$ and $\bigcap_i Y_i$?

8. Prove that if R and S are relations, and $Q = R|S$, then $\breve{Q} = \breve{S}|\breve{R}$. Prove that if $F : X \to Y$ and $G : Y \to Z$ bijectively, then $(G \circ F)^{-1} = F^{-1} \circ G^{-1}$.

9. Prove that the composition of two injections is an injection. Do the same for surjections and bijections.

10. Prove that if R and S are relations, then $(R|S)[A] = S[R[A]]$. Prove that if $F : X \to Y$ and $G : Y \to Z$ then $(G \circ F)[A] = G[F[A]]$.

11. Prove that if R is a relation, then $(X \times Y)|R = X \times R[Y]$. What's $R|(X \times Y)$?

12. Prove all the distributive laws mentioned.

13. Prove the de Morgan laws.

14. Prove the *modular law* $X \subseteq Z \Leftrightarrow X \cup (Y \cap Z) = (X \cup Y) \cap Z$.
Prove that $X \cup Y = X \cup Z$ & $X \cap Y = X \cap Z \Rightarrow Y = Z$.

15. a. Prove $\bigcup \mathcal{P} X = X$.
b. Prove $\mathcal{X} \subseteq \mathcal{P} \bigcup \mathcal{X}$.
c. Find $\mathcal{X}$ so that $\mathcal{X} = \mathcal{P} \bigcup \mathcal{X}$.
d. Find $\mathcal{X}$ so that $\mathcal{X} \neq \mathcal{P} \bigcup \mathcal{X}$.

16. When does a relative complement of a set equal that set itself?

17. a. Why can't we define $\bigcap \phi$?
b. Why can't we define an *absolute* complement $-X = \{t : t \notin X\}$?

18. a. Prove that if R is a relation, then $R[\bigcup_i A_i] = \bigcup_i R[A_i]$.

Suppose S is a relation and for each i, R_i is a relation. Prove

b. $\operatorname{Dom} \bigcup_i R_i = \bigcup_i \operatorname{Dom} R_i$ and $\operatorname{Rng} \bigcup_i R_i = \bigcup_i \operatorname{Rng} R_i$
c. $R = \bigcup_i R_i \Rightarrow \breve{R} = \bigcup_i \breve{R}_i$
d. $(\bigcup_i R_i)|S = \bigcup_i (R_i|S)$ and $S|\bigcup_i R_i = \bigcup_i (S|R_i)$
e. $(\bigcup_i R_i)[A] = \bigcup_i R_i[A]$.

19. If $F : X \to Y$, then define $\breve{F}“ : \mathcal{P}Y \to \mathcal{P}X$ by setting $\breve{F}“(B) = \breve{F}[B]$ for every $B \in \mathcal{P}Y$. Prove that if F is injective, then $\breve{F}“$ is surjective. Prove that if F is surjective, then $\breve{F}“$ is injective.

Substantial problems

1. Undertake routine exercise 18 with unions replaced by intersections. You'll find that you must replace many equations by inclusions. In those cases, find examples where the equations hold, and examples where they don't. Keep the examples simple—use intersections of two sets only.

2. Let R be a relation. Prove that

$$B \subseteq \operatorname{Rng} R \Rightarrow B \subseteq R[\breve{R}[B]].$$

Prove that R is a function if and only if

$$\forall B [B \subseteq \operatorname{Rng} R \Rightarrow B = R[\breve{R}[B]]].$$

Find a function F and a set $B \subseteq \operatorname{Dom} F$ such that $B \neq \breve{F}[F[B]]$, and another F and B such that the equation does hold.

3. A function $F : X \to Y$ is called *right cancellative* if

$$G : Y \to Z \ \& \ G' : Y \to Z' \ \& \ G \circ F = G' \circ F \ \Rightarrow \ G = G'.$$

It's *left cancellative* if

$$E : W \to X \ \& \ E' : W' \to X \ \& \ F \circ E = F \circ E' \ \Rightarrow \ E = E'.$$

Prove that F is injective if and only if it's left cancellative and surjective if and only if it's right cancellative.

4. Suppose that for each $i \in I$, F_i is a function from a subset of a set X to a set Y. Further, assume $(\forall i, j \in I)(\exists k \in I)[F_i \subseteq F_k \ \& \ F_j \subseteq F_k]$. Prove that $\bigcup_i F_i$ is a function from a subset of X to Y, and if each F_i is injective, then so is $\bigcup_i F_i$.

5. Consider some sets A_n for $n \in \mathbb{N}$. Define

$$\text{Liminf}_n A_n = \bigcup_m \bigcap_{n \ge m} A_n \qquad \text{Limsup}_n A_n = \bigcap_m \bigcup_{n \ge m} A_n .$$

Prove

a. $\text{Liminf}_n A_n \subseteq \text{Limsup}_n A_n$
b. $\forall n [A_n \subseteq A_{n+1}] \Rightarrow \text{Liminf}_n A_n = \bigcup_n A_n = \text{Limsup}_n A_n$
c. $\forall n [A_{n+1} \subseteq A_n] \Rightarrow \text{Liminf}_n A_n = \bigcap_n A_n = \text{Limsup}_n A_n$.

6. Let $m, n \in \mathbb{N}$ and X and Y be sets with m and n elements. How many elements have the sets

$$(X \times \{0\}) \cup (Y \times \{1\}) \qquad X \times Y \qquad \mathcal{P}X \qquad Y^X?$$

How many injections are there from X to Y? How many bijections are there from X to X?

7. Show that this condition on t has no extension: $\neg\exists s [s \in t \ \& \ t \in s]$.

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